Two Results on Bounding the Roots of Interval Polynomials

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Abstract

The problem of bounding the zeros of a polynomial of degree $n \geq 2$, with real or complex coefficients, is considered and an improved Cauchy bound is proposed. This new bound is applied to two examples and compared with existing approaches. The methodology is then extended to the case of polynomials of degree $n \geq 2$ with interval coefficients. A counter-example to Theorem 3.8 in [2], used in the αBB algorithm to compute a tight lower bound on the minimum real part of the zeros of an interval polynomial is also presented. An alternative approach which uses the improved Cauchy bound as a starting point is developed.

1 Introduction

The ability to determine the roots of polynomials gives access to crucial information in the analysis of many chemical engineering systems. An area of particular interest is global optimization, which has numerous applications in chemical engineering, from the characterization of phase equilibrium to process synthesis and the enclosure of all solutions of nonlinear systems of equations. In this context, the convexity of functions, or lack thereof, is of prime importance. It can be determined by examining the eigenvalues of the Hessian matrix of a functions, or equivalently the roots of the characteristic polynomial of the Hessian matrix. In the following, we describe an improvement over the classical Cauchy bound for the maximal modulus of a complex polynomial of degree n [5]. We extend the application of this bound to the case of interval polynomials. We then show how this bound can be combined with the Kharitonov theorem [9] to obtain a bound on the minimum root of a characteristic polynomial with negative roots. This approach can be embedded within the αBB global optimization algorithm [2, 3] where it enables the construction of rigorously valid convex underestimators.

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2 Improved Cauchy bound

2.1 Improved Cauchy bound for polynomials with real or complex coefficients

2.1.1 Background

Let

$$\mathcal{P}(\lambda) = \lambda^n + \sum_{k=1}^{n} a_k \lambda^{n-k} \tag{1}$$

be a polynomial of degree n with arbitrary complex coefficients a_k, $k = 1, \ldots, n$. Let λ_k, $k = 1, \cdots, n$, denote the zeros of $\mathcal{P}(\lambda)$ and ρ their maximal modulus. The problem of finding bounds for ρ has been recently studied in [12] and [13], see also references therein. The computational complexity of the methods proposed in [12, 13] is high, i.e. $O(n^2)$ and $O(n^3)$, respectively. These two methods are based on the following Cauchy bound.

Theorem 2.1 *(Cauchy [5], see also Kurosh [10] for a more accessible proof.) The zeros of $\mathcal{P}(\lambda)$ satisfy*

$$\rho = \max_{k=1,\cdots,n} |\lambda_k| < A + 1 \tag{2}$$

where

$$A = \max_{k=1,\cdots,n} |a_k|. \tag{3}$$

The computational complexity of Cauchy's bound is $O(n)$, mainly comparisons. We derive an improved Cauchy bound which preserves its $O(n)$ computational complexity. We then present two examples which demonstrate the improvements obtained by using the proposed bound.

2.1.2 The improved bound

Theorem 2.2 *For $n \geq 2$,*

$$\rho = \max_{k=1,\cdots,n} |\lambda_k| < r^+, \tag{4}$$

where

$$r^+ = 0.5(1 + |a_1|) + 0.5\sqrt{(1 + |a_1|)^2 - 4(|a_1| - B)}, \tag{5}$$

is the largest positive solution of the quadratic equation

$$r^2 - r(1 + |a_1|) + |a_1| - B = 0 \tag{6}$$

and

$$B = \max_{k=2,\cdots,n} |a_k|. \tag{7}$$

Proof We will show that if $|\lambda| \geq r^+$, λ cannot be a root of $\mathcal{P}(\lambda)$. Let

$$R(\lambda) = \sum_{k=2}^{n} a_k \lambda^{n-k}. \tag{8}$$

Let $r = |\lambda|$. Assuming that $r > 1$, we obtain

$$\begin{aligned} |R(\lambda)| &\leq \sum_{k=2}^{n} |a_k||\lambda^{n-k}| && (9) \\ &\leq B \sum_{k=2}^{n} |\lambda^{n-k}| && (10) \\ &\leq B \frac{|\lambda^{n-1}| - 1}{|\lambda| - 1} && (11) \\ &< B \frac{|\lambda^{n-1}|}{|\lambda| - 1}. && (12) \end{aligned}$$

Assuming in addition that $r = |\lambda| > |a_1|$, we find

$$|\lambda^n + a_1 \lambda^{n-1}| = r^{n-1} |\lambda + a_1| \geq r^{n-1}(r - |a_1|). \tag{13}$$

Combining Eqs. (12) and (13), we obtain that if

$$(r - |a_1|)(r - 1) \geq B \Leftrightarrow r^2 - (1 + |a_1|)r + |a_1| - B \geq 0 \tag{14}$$

then,

$$\left|\lambda^n + a_1 \lambda^{n-1}\right| > \left|\sum_{k=2}^{n} a_k \lambda^{n-k}\right|. \tag{15}$$

Before completing the proof, note that the discriminant of the parabola in Eq. (14) attains its minimal value, $||a_1| - 1|$, for $B = 0$. Therefore, its zeros are always real. Hence, since Eq. (14) is a minimal parabola, with r^+ its largest positive zero, $|\lambda| \geq \max\{1, |a_1|, r^+\}$ implies Eq. (15). In addition, $r^+ \geq \max\{1, |a_1|\}$, as can be shown from the definition of r^+, and so $|\lambda| \geq r^+$ implies Eq. (15). In other words, ρ must satisfy Eq. (4) and the proof is completed. ∎

It remains to show that the modified Cauchy bound (Theorem 2.2) is always better than Cauchy's bound. By combining Eqs. (2) and (4) one has to show that

$$1 + A \geq r^+. \tag{16}$$

To show that Eq. (16) holds true, we solve the inequality $r^+ \leq A + 1$ noting that $A \geq B$. We find that the inequality is satisfied if $A \geq |a_1|$ which is always true and proves that the new bound is always better than the Cauchy bound.

The improvement on the Cauchy bound arises from the use of an additional term from the polynomial under consideration. It can be expected that as more terms are taken into account, greater accuracy is achieved.

2.1.3 Examples

Example 1 (Zeheb [12])
Let

$$\mathcal{P}(\lambda) = \lambda^3 + 3\lambda^2 + 2\lambda + 1. \tag{17}$$

Applying Theorem 2.1, Cauchy's bound produces $\rho \leq 4$, whereas applying Theorem 2.2, we obtain the bound $\rho \leq 3.732$. The following bounds for this example appear in [13]:

Method	Bound	Method	Bound
Zeheb [12]	2.750	Marden [11, Theorem 27.3]	3.627
Marden [11, Eq. (27.19)]	3.873	Marden [11, Eq. (27.25)]	2.828
Marden [11, Eq. (27.26)]	5.414	Joyal *et al.* [8, Theorem 1]	4.000
Joyal *et al.* [8, Theorem 1 (corollary 2)]	3.193	Joyal *et al.* [8, Theorem 1 (corollary 3)]	3.000
Datt and Govil [6, Theorem 1]	3.951	Datt and Govil [6, Theorem 2]	3.953
Boese and Luther [4]	3.951		

The authors in [13] transformed the polynomial $\mathcal{P}(\lambda)$ into the polynomial

$$\mathcal{Q}(\lambda) = \lambda^3 - 5\lambda^2 - 2\lambda - 1 \tag{18}$$

whose zeros are the square of the zeros of $\mathcal{P}(\lambda)$. Applying Cauchy's bound, they obtained the bound $\rho \leq \sqrt{1+5} = 2.45$. Now, by applying to $\mathcal{Q}(\lambda)$ the modified Cauchy bound we obtain the improved bound $\rho = 2.325$. Note that the exact bound is $\rho = 2.3247$.

Example 2 Let

$$\mathcal{P}(\lambda) = \lambda^5 + 15\lambda^4 + 85\lambda^3 - 225\lambda^2 + 274\lambda - 120. \tag{19}$$

Note that the zeros of $\mathcal{P}(\lambda)$ are $1, \cdots, 5$, therefore $\rho = 5$.

By Theorem 2.1, Cauchy's bound produces $\rho \leq 275$, whereas Theorem 2.2 produces the bound $\rho \leq 25.9722$. Here, Zeheb's bound [12] produces $\rho \leq 144.3148$.

To find the bound of [13], one has to transform $\mathcal{P}(\lambda)$ into

$$\mathcal{Q}(\lambda) = \lambda^5 - 55\lambda^4 + 1023\lambda^3 - 7645\lambda^2 + 21076\lambda - 14400 \tag{20}$$

whose zeros are the square of the zeros of $\mathcal{P}(\lambda)$, i.e., 1, 4, 9, 16 and 25. Applying Theorem 2.1 to $\mathcal{Q}(\lambda)$, we obtain that $\rho \leq \sqrt{1 + 21076} = 145.1792$, whereas using the proposed bound, Theorem 2.2, we obtain the improved bound $\rho \leq \sqrt{175.6652} = 13.2539$.

2.2 Improved Cauchy bound for interval polynomials

We first extend the Cauchy bound for the zeros of complex polynomials to interval polynomials. We then derive a modified and tighter bound for this class of polynomials. Given a twice continuously differentiable function $f(\mathbf{x})$, $\mathbf{x} \in X$, we consider the following characteristic interval polynomial derived from its interval Hessian matrix over the domain X:

$$\mathcal{P}_{f,X}(\lambda) = \lambda^n + \sum_{k=1}^{n} [\underline{a}_k, \overline{a}_k] \lambda^{n-k},$$

with $n \geq 1$, and $\underline{a}_k, \overline{a}_k$ real numbers such that $\underline{a}_k \leq \overline{a}_k$ for all k. This type of polynomial arises in [2], for example.

For two real intervals $[\underline{b}, \overline{b}]$ and $[\underline{c}, \overline{c}]$, we define the following relation:

$$[\underline{b}, \overline{b}] \geq [\underline{c}, \overline{c}] \Leftrightarrow \underline{b} \geq \overline{c}.$$

We also define the notation $|a|_k = \max\{|\underline{a}_k|, |\overline{a}_k|\}$.

2.2.1 Cauchy bound for interval polynomials

We follow the proof of the Cauchy bound theorem by Kurosh [10] and extend it to the case of the interval polynomial $\mathcal{P}_{f,X}(\lambda)$. First, we show that the modulus of the highest degree term is strictly greater than the modulus of the sum of all the other terms for sufficiently large values of λ.

Lemma 2.1 *If $|\lambda| \geq \mathcal{A} + 1$, with $\mathcal{A} = \max_{k=1,\cdots,n} |a|_k$, we have*

$$|\lambda^n| > \left| \sum_{k=1}^{n} [\underline{a}_k, \overline{a}_k] \lambda^{n-k} \right|. \tag{21}$$

Proof

$$\left|\sum_{k=1}^{n}[\underline{a}_k,\overline{a}_k]\lambda^{n-k}\right| \leq \sum_{k=1}^{n}\left(|a|_k|\lambda^{n-k}|\right) \leq \mathcal{A}\sum_{k=1}^{n}|\lambda^{n-k}| = \mathcal{A}\tfrac{|\lambda^n|-1}{|\lambda|-1}. \tag{22}$$

For $|\lambda|>1$,

$$\mathcal{A}\frac{|\lambda^n|-1}{|\lambda|-1} < \mathcal{A}\frac{|\lambda^n|}{|\lambda|-1}. \tag{23}$$

Therefore,

$$\left|\sum_{k=1}^{n}[\underline{a}_k,\overline{a}_k]\lambda^{n-k}\right| < \mathcal{A}\frac{|\lambda^n|}{|\lambda|-1}. \tag{24}$$

Thus, Eq. (21) holds when $|\lambda^n| \geq \mathcal{A}\frac{|\lambda^n|}{|\lambda|-1}$ and $|\lambda|>1$, that is, when $|\lambda| \geq \mathcal{A}+1$. ∎

Theorem 2.3 *The zeros, λ_k, of the interval polynomial $\mathcal{P}_{f,X}(\lambda)$ are such that*

$$\rho = \max_{k=1,\cdots,n}|\lambda_k| < \mathcal{A}+1, \tag{25}$$

with $\mathcal{A} = \max_{k=1,\cdots,n}|a|_k$.

Proof This theorem follows directly from Lemma 2.1. ∎

2.2.2 Extension of the improved Cauchy bound to interval polynomials

We now derive an alternative bound on the maximal modulus of $\mathcal{P}_{f,X}(\lambda)$ and show that it is at least as good as the extended Cauchy bound.

Theorem 2.4 *For $n \geq 2$, the maximal modulus ρ of $\mathcal{P}_{f,X}(\lambda)$ is such that*

$$\rho < r^+, \tag{26}$$

where

$$r^+ = 0.5(1+|a|_1) + 0.5\sqrt{(1+|a|_1)^2 - 4(|a|_1-\mathcal{B})}, \tag{27}$$

and

$$\mathcal{B} = \max_{k=2,\cdots,n}|a|_k. \tag{28}$$

Proof Let

$$R(\lambda) = \sum_{k=2}^{n} [\underline{a}_k, \overline{a}_k]\lambda^{n-k}. \tag{29}$$

Assuming $r = |\lambda| > 1$, we obtain

$$\begin{aligned} |R(\lambda)| &\leq \sum_{k=2}^{n} |a|_k |\lambda^{n-k}| & (30)\\ &\leq \mathcal{B}\sum_{k=2}^{n} |\lambda^{n-k}| & (31)\\ &\leq \mathcal{B}\frac{|\lambda^{n-1}|-1}{|\lambda|-1} & (32)\\ &< \mathcal{B}\frac{|\lambda^{n-1}|}{|\lambda|-1}. & (33)\end{aligned}$$

Assuming in addition that $r = |\lambda| > |a|_1$, we find

$$|\lambda^n + [\underline{a}_1, \overline{a}_1]\lambda^{n-1}| = r^{n-1}|\lambda + [\underline{a}_1, \overline{a}_1]| \geq r^{n-1}(r - |a|_1). \tag{34}$$

Combining Eqs. (33) and (34), we can show that if

$$(r - |a|_1)(r-1) \geq \mathcal{B}, \tag{35}$$

or equivalently,

$$r^2 - (1 + |a|_1)r + |a|_1 - \mathcal{B} \geq 0 \tag{36}$$

then,

$$\left|\lambda^n + [\underline{a}_1, \overline{a}_1]\lambda^{n-1}\right| > \left|\sum_{k=2}^{n} [\underline{a}_k, \overline{a}_k]\lambda^{n-k}\right|. \tag{37}$$

The minimum value of the discriminant of the parabola in Eq. (36) occurs at $\mathcal{B} = 0$ and is equal to $|1 - |a|_1|$. Therefore, its zeros are all real. Since Eq. (36) is a minimal parabola, the condition in Eq. (36) is satisfied for $r = |\lambda| \geq r^+$, the largest root of parabola (36) given by Eq. (27). Since, when Eq. (37) is true, $\mathcal{P}_{f,X}(\lambda)$ cannot have any zeros and $r^+ \geq \max\{1, |a|_1\}$, r^+ is a bound on the maximal modulus of $\mathcal{P}_{f,X}(\lambda)$. ∎

Corollary 2.1 *$b_2 = r^+$, the bound given by Theorem 2.4, is at least as good as the Cauchy bound, $b_1 = \mathcal{A} + 1$.*

Proof Since $0 \leq \mathcal{B} \leq \mathcal{A}$, $r^+ \leq 0.5(1 + |a|_1) + 0.5\sqrt{(1+|a|_1)^2 - 4(|a|_1 - \mathcal{A})}$. $0.5(1 + |a|_1) + 0.5\sqrt{(1+|a|_1)^2 - 4(|a|_1 - \mathcal{A})} - (\mathcal{A}+1)$ is a monotonically increasing function of $|a|_1$, so its maximum value occurs at $|a|_1 = \mathcal{A}$ and is equal to 0. Therefore, $b_2 \leq b_1$.
∎

3 The Kharitonov theorem in the αBB algorithm

3.1 Counter-example to Theorem 3.8 of [2]

In [2, Theorem 3.8], see also [1], the authors presented the following extension of Kharitonov's theorem [9].

Theorem 3.1 *Let an interval polynomial family $\mathcal{P}_{f,X}(\lambda)$ be defined by*

$$\mathcal{P}_{f,X}(\lambda) = [\underline{a}_n, \overline{a}_n] + [\underline{a}_{n-1}, \overline{a}_{n-1}]\lambda + \cdots + [\underline{a}_1, \overline{a}_1]\lambda^{n-1} + \lambda^n, \tag{38}$$

where $\underline{a}_i \leq \overline{a}_i, \ \forall i$.
Let $\mathcal{P}_4(\lambda)$ denote the subset of this family containing the following four real polynomials (Kharitonov polynomials):

$$\begin{aligned}
K_1(\lambda) &= \underline{a}_n + \underline{a}_{n-1}\lambda + \overline{a}_{n-2}\lambda^2 + \overline{a}_{n-3}\lambda^3 + \underline{a}_{n-4}\lambda^4 + \underline{a}_{n-5}\lambda^5 + \overline{a}_{n-6}\lambda^6 + \cdots \\
K_2(\lambda) &= \overline{a}_n + \overline{a}_{n-1}\lambda + \underline{a}_{n-2}\lambda^2 + \underline{a}_{n-3}\lambda^3 + \overline{a}_{n-4}\lambda^4 + \overline{a}_{n-5}\lambda^5 + \underline{a}_{n-6}\lambda^6 + \cdots \\
K_3(\lambda) &= \overline{a}_n + \underline{a}_{n-1}\lambda + \underline{a}_{n-2}\lambda^2 + \overline{a}_{n-3}\lambda^3 + \overline{a}_{n-4}\lambda^4 + \underline{a}_{n-5}\lambda^5 + \underline{a}_{n-6}\lambda^6 + \cdots \\
K_4(\lambda) &= \underline{a}_n + \overline{a}_{n-1}\lambda + \overline{a}_{n-2}\lambda^2 + \underline{a}_{n-3}\lambda^3 + \underline{a}_{n-4}\lambda^4 + \overline{a}_{n-5}\lambda^5 + \overline{a}_{n-6}\lambda^6 + \cdots .
\end{aligned}$$

Then $\mathcal{P}_{f,X}(\lambda)$ and $\mathcal{P}_4(\lambda)$ have the same $\Re(\lambda)_{min}$, where $\Re(\lambda)_{min}$ is the smallest real part of the roots of $\mathcal{P}_{f,X}(\lambda)$.

We present a counter-example which was found via a computer search. Let

$$\mathcal{P}_{f,X}(\lambda) = [0.15, 7.11] + [1.52, 6.21]\lambda + [0.20, 2.35]\lambda^2 + [1.14, 3.15]\lambda^3 + [1.20, 2.73]\lambda^4 + \lambda^5 \tag{39}$$

Its four Kharitonov polynomials are:

$$K_1(\lambda) = 0.15 + 1.52\lambda + 2.35\lambda^2 + 3.15\lambda^3 + 1.20\lambda^4 + \lambda^5, \tag{40}$$

$$K_2(\lambda) = 7.11 + 6.21\lambda + 0.20\lambda^2 + 1.14\lambda^3 + 2.73\lambda^4 + \lambda^5, \tag{41}$$

$$K_3(\lambda) = 7.11 + 1.52\lambda + 0.20\lambda^2 + 3.15\lambda^3 + 2.73\lambda^4 + \lambda^5, \tag{42}$$

$$K_4(\lambda) = 0.15 + 6.21\lambda + 2.35\lambda^2 + 1.14\lambda^3 + 1.20\lambda^4 + \lambda^5. \tag{43}$$

Now, consider the following counter example:

$$E(\lambda) = 5.82 + 1.87\lambda + 1.96\lambda^2 + 1.19\lambda^3 + 2.70\lambda^4 + \lambda^5 \in \mathcal{P}_{f,X}(\lambda). \tag{44}$$

Using MatLab we obtain

$$-1.7350 \leq \min_{1\leq \ell \leq 4} \Re(\lambda(K_\ell)) \leq 0.8579 \tag{45}$$

and

$$\min \Re(\lambda(E)) = -2.5585, \tag{46}$$

where $\lambda(K_\ell)$ and $\lambda(E)$ denote the sets of eigenvalues of the polynomials $K_\ell(\lambda)$ and $E(\lambda)$, respectively, and $\Re$ denotes the real part. Hence, using $\Re(\lambda)_{min}$ as defined in Theorem 3.1 we obtain

$$\Re(\lambda)_{min} = \min_{P(\lambda)\in \mathcal{P}_{f,X}(\lambda)} \Re(\lambda(P)) \leq \min \Re(\lambda(E)) < \min_{1\leq \ell \leq 4} \Re(\lambda(K_\ell)), \tag{47}$$

thus rendering Theorem 3.1 incorrect.

3.2 Alternative approach

In light of the above counter-example, we modify the results of Theorem 3.1 (Theorem 3.8 of [2]) and derive the correct form of the Kharitonov polynomials that must be used to compute $\Re(\lambda)_{min}$. We then show how this result and the improved Cauchy bound for interval polynomials can be used in the αBB algorithm.

The identification of $\Re(\lambda)_{min}$ is based on the following theorem.

Theorem 3.2 *Consider the interval polynomial family*

$$\mathcal{P}_{f,X}(\lambda) = [\underline{a}_n, \overline{a}_n] + [\underline{a}_{n-1}, \overline{a}_{n-1}]\lambda + \cdots + [\underline{a}_1, \overline{a}_1]\lambda^{n-1} + \lambda^n, \tag{48}$$

which contains at least one polynomial whose zeros are not all in the right-half plane. A given negative real number, λ_*, is a lower bound on $\Re(\lambda)_{min}$ if and only if all the zeros of the interval polynomial

$$\mathcal{Q}_{f,X,\lambda_*}(\lambda) = \mathcal{P}_{f,X}(\lambda + \lambda_*) = [\underline{a}_n, \overline{a}_n] + [\underline{a}_{n-1}, \overline{a}_{n-1}](\lambda + \lambda_*) + \cdots + (\lambda + \lambda_*)^n \tag{49}$$

lie in the right-half plane.

Proof Let $\Lambda^P = \{\lambda \in \mathbb{C} : \mathcal{P}_{f,X}(\lambda) = 0\}$ and $\Lambda^Q = \{\lambda \in \mathbb{C} : \mathcal{Q}_{f,X,\lambda_*}(\lambda) = 0\}$. From the definition of $\mathcal{Q}_{f,X,\lambda_*}(\lambda)$,

$$\lambda \in \Lambda^P \Leftrightarrow (\lambda - \lambda_*) \in \Lambda^Q \tag{50}$$

Let $\lambda_* \leq \Re(\lambda)_{min} \leq 0$. By definition, $\Re(\lambda) - \Re(\lambda)_{min} \geq 0$ for all $\lambda \in \Lambda^P$. Therefore, $\Re(\lambda) - \lambda_* \geq 0$, $\forall \lambda \in \Lambda^P$.. Hence,

$$\Re(\lambda - \lambda_*) \geq 0, \; \forall (\lambda - \lambda_*) \in \Lambda^Q, \tag{51}$$

that is, all the zeros of $\mathcal{Q}_{f,X,\lambda_*}(\lambda)$ lie in the right-half plane.

If $\Re(\lambda - \lambda_*) \geq 0$, $\forall(\lambda - \lambda_*) \in \Lambda^Q$, then $\Re(\lambda) \geq \lambda_*$, $\forall(\lambda - \lambda_*) \in \Lambda^Q$. Using Eq. (50), we conclude that λ_* is a lower bound on $\Re(\lambda)_{min}$. ∎

While the Kharitonov polynomials of $\mathcal{P}_{f,X}(\lambda)$ were used in [1, 2], the polynomials of $\mathcal{Q}_{f,X,\lambda_*}(\lambda)$ should instead be used. These two sets of Kharitonov polynomials differ since λ_* appears in the coefficients of $\mathcal{Q}_{f,X,\lambda_*}(\lambda)$. Expanding the above expression for $\mathcal{Q}_{f,X,\lambda_*}(\lambda)$ yields

$$\mathcal{Q}_{f,X,\lambda_*}(\lambda) = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \lambda_*^{n-k}\lambda^k + \sum_{i=0}^{n-1}[\underline{a}_{n-i}, \overline{a}_{n-i}] \sum_{k=0}^{i} \begin{pmatrix} i \\ k \end{pmatrix} \lambda_*^{i-k}\lambda^k \tag{52}$$

where $\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 1$.

Grouping terms with the same power of λ, we get

$$\mathcal{Q}_{f,X,\lambda_*}(\lambda) = \lambda^n + \sum_{k=0}^{n-1} \left(\sum_{i=k}^{n} [\underline{a}_{n-i}, \overline{a}_{n-i}] \begin{pmatrix} i \\ k \end{pmatrix} \lambda_*^{i-k} \right) \lambda^k \tag{53}$$

where, based on the definition of $\mathcal{P}_{f,X,\lambda_*}(\lambda)$, $[\underline{a}_0, \overline{a}_0] = 1$.

We note that we are only interested in finding λ_* if $\Re(\lambda)_{min}$ is negative, that is, if at least one root of a Kharitonov polynomial of $\mathcal{P}_{f,X}(\lambda)$ lies in the left half-plane. Thus, assuming that λ_* is negative, we can get the interval coefficients of $\mathcal{Q}_{f,X,\lambda_*}(\lambda)$, given by

$$\mathcal{Q}_{f,X,\lambda_*}(\lambda) = \lambda^n + \sum_{k=0}^{n-1} \sum_{i=k}^{n} [\underline{c}_{ik}, \overline{c}_{ik}]\lambda^k, \tag{54}$$

where

$$\underline{c}_{ik} = \begin{cases} \begin{pmatrix} i \\ k \end{pmatrix} \lambda_*^{i-k}\underline{a}_{n-i}, & \text{if } i-k \text{ even} \\ \begin{pmatrix} i \\ k \end{pmatrix} \lambda_*^{i-k}\overline{a}_{n-i}, & \text{if } i-k \text{ odd} \end{cases} \tag{55}$$

$$\overline{c}_{ik} = \begin{cases} \begin{pmatrix} i \\ k \end{pmatrix} \lambda_*^{i-k}\overline{a}_{n-i}, & \text{if } i-k \text{ even} \\ \begin{pmatrix} i \\ k \end{pmatrix} \lambda_*^{i-k}\underline{a}_{n-i}, & \text{if } i-k \text{ odd} \end{cases} \tag{56}$$

The coefficients of $\mathcal{Q}_{f,X,\lambda_*}(\lambda)$ are thus given by polynomials in λ_*. The Kharitonov polynomials of $\mathcal{Q}_{f,X,\lambda_*}(\lambda)$ can now be derived using the standard formula:

$$K_{\mathcal{Q},\lambda_*,1}(\lambda) = \sum_{i=0}^{n} \underline{c}_{i0} + \sum_{i=1}^{n} \underline{c}_{i1}\lambda + \sum_{i=2}^{n} \overline{c}_{i2}\lambda^2 + \sum_{i=3}^{n} \overline{c}_{i3}\lambda^3 + \sum_{i=4}^{n} \underline{c}_{i4}\lambda^4 + \cdots + \lambda^n \tag{57}$$

$$K_{\mathcal{Q},\lambda_*,2}(\lambda) = \sum_{i=0}^{n} \overline{c}_{i0} + \sum_{i=1}^{n} \overline{c}_{i1}\lambda + \sum_{i=2}^{n} \underline{c}_{i2}\lambda^2 + \sum_{i=3}^{n} \underline{c}_{i3}\lambda^3 + \sum_{i=4}^{n} \overline{c}_{i4}\lambda^4 + \cdots + \lambda^n \tag{58}$$

$$K_{\mathcal{Q},\lambda_*,3}(\lambda) = \sum_{i=0}^{n} \overline{c}_{i0} + \sum_{i=1}^{n} \underline{c}_{i1}\lambda + \sum_{i=2}^{n} \underline{c}_{i2}\lambda^2 + \sum_{i=3}^{n} \overline{c}_{i3}\lambda^3 + \sum_{i=4}^{n} \overline{c}_{i4}\lambda^4 + \cdots + \lambda^n \tag{59}$$

$$K_{\mathcal{Q},\lambda_*,4}(\lambda) = \sum_{i=0}^{n} \underline{c}_{i0} + \sum_{i=1}^{n} \overline{c}_{i1}\lambda + \sum_{i=2}^{n} \overline{c}_{i2}\lambda^2 + \sum_{i=3}^{n} \underline{c}_{i3}\lambda^3 + \sum_{i=4}^{n} \underline{c}_{i4}\lambda^4 + \cdots + \lambda^n \tag{60}$$

These Kharitonov polynomials replace those used in Theorem 3.1 of [2] to yield the following theorem:

Theorem 3.3 *If $\mathcal{P}_{f,X}(\lambda)$ has at least one zero in the left-half plane, λ_* is a lower bound on $\Re(\lambda)_{min}$ if and only if all the zeros of the Kharitonov polynomials $K_{\mathcal{Q},\lambda_*,j}$, $j = 1, \cdots, n$, defined by Eqs. (57-60), lie in the right-half plane.*

This theorem can be combined with the improved Cauchy bound for interval polynomials to give an arbitrarily tight bound on $\Re(\lambda)_{min}$. Since the improved Cauchy bound does not provide information on the sign of the roots, we start by applying the Kharitonov theorem to $\mathcal{P}_{f,X}(\lambda)$ to ascertain whether at least one of its zeros is negative. If so, we then use bisection to refine the first estimate of a lower bound on $\Re(\lambda)_{min}$ given by Theorem 2.4.

Step 1 Check whether at least one of the Kharitonov polynomials of $\mathcal{P}_{f,X}(\lambda)$ has a negative root. If all roots are in the right-half plane, terminate: the function is convex.

Step 2 Set $i = 1$, $\lambda^L = -b_2$, $\lambda^U = 0$. Set $l_1 = \lambda^L/2$. Set tolerance ϵ.

Step 3 Compute the real coefficients of $K_{\mathcal{Q},l_i,j}(\lambda)$, for $j = 1, \ldots, 4$.

Step 4 If all the zeros of these four Kharitonov polynomials lie in the right half-plane, l_i is a lower bound on $\Re(\lambda)_{min}$: set $\lambda^L = l_i$. Go to Step 6.

Step 5 Otherwise, set $\lambda^U = l_i$.

Step 6 Set $i = i + 1$. If $\lambda^U - \lambda^L \leq \epsilon$, set $\lambda_* = \lambda^L$ and terminate. Else, set $l_i = \frac{\lambda^U + \lambda^L}{2}$ and go back to Step 3.

This procedure is exemplified on the counter-example presented above. In this case, the interval polynomial $\mathcal{Q}_{f,X,\lambda_*}(\lambda)$ is given by

$$\begin{aligned}\mathcal{Q}_{f,X,\lambda_*}(\lambda) = {} & \lambda^5 + \{[1.20, 2.73] + 5\lambda_*\}\,\lambda^4 + \left\{[1.14, 3.15] + [4.80, 10.92]\lambda_* + 10\lambda_*^2\right\}\lambda^3 \\ & + \left\{[0.20, 2.35] + [3.42, 9.45]\lambda_* + [7.20, 16.38]\lambda_*^2 + 10\lambda_*^3\right\}\lambda^2 \\ & + \left\{[1.52, 6.21] + [0.40, 4.70]\lambda_* + [3.42, 9.45]\lambda_*^2 + [4.80, 10.92]\lambda_*^3 + 5\lambda_*^4\right\}\lambda \\ & + \left\{[0.15, 7.11] + [1.52, 6.21]\lambda_* + [0.20, 2.35]\lambda_*^2 + [1.14, 3.15]\lambda_*^3 + [1.20, 2.73]\lambda_*^4 + \lambda_*^5\right\}.\end{aligned} \tag{61}$$

To get an initial guess for λ_*, we note that $\mathcal{A} = \mathcal{B} = 7.11$ and $|a|_1 = 2.73$. Thus, $b_2 = 4.6683$. We set $\lambda^L = -4.6683$ and $l_1 = -2.33415$. We find that all the Kharitonov polynomials of $\mathcal{Q}_{f,X,-2.33415}(\lambda)$ have at least one zero with a negative real part. Hence, l_1 is an upper bound on $\Re(\lambda)_{min}$. We set $\lambda^U = -2.33415$. A new guess is given by $l_2 = -3.50123$. In this case, the Kharitonov polynomials of $\mathcal{Q}_{f,X,-3.50123}(\lambda)$ are all anti-stable, that is, all their zeros lie in the open right-half plane. Hence, $\lambda^L = -3.50123$ and $l_3 = -2.91769$. We may continue this procedure until the desired accuracy is obtained. For instance, after 7 iterations, we have $-3.35535 \leq \Re(\lambda)_{min} \leq -3.31888$ and we may set $\lambda_* = -3.35535$.

The procedure presented here for the evaluation of a valid lower bound on the minimum real part of the roots of an interval polynomial is more computationally intensive than that based on Theorem 3.1. The usefulness of Theorem 3.1 for the derivation of underestimators within the αBB algorithm has been studied in [1, 2]. In the light of the counter-example, we tested the validity of the underestimators obtained using Theorem 3.1 for the examples previously addressed, by comparing them with those derived rigorously using Hertz's method of computing minimum eigenvalues. In all the problems studied in [1, 2, 3], the underestimators were found to be strictly valid.

As reported in [3], computational experience within the αBB algorithm has shown the use of the Kharitonov polynomials to be a very costly process in terms of number of iterations and CPU time. Not only is the calculation of the coefficients of the interval polynomial intensive, but the resulting α values lead to relatively loose underestimators, therefore slowing down the progress of the algorithm. These conclusions are accentuated by the iterative nature of the revised approach. Significantly more successful techniques, such as a modification of the Gerschgorin theorem [7], have been proposed in [3, Method II.1].

4 Conclusions

We derived a new bound on the maximal modulus of the zeros of the polynomial with real or complex coefficients. This bound is at least as good as the Cauchy bound, is computationally efficient ($O(n)$), and was shown on two examples to compare well to other approaches. We also derived the correct Kharitonov polynomials which allow us to obtain an arbitrarily tight lower bound on the minimum real part of the zeros of an interval polynomial. This is achieved through an iterative procedure which relies on an extension of the improved Cauchy bound to interval polynomials.

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