

# A Decomposition-Based Global Optimization Approach for Solving Bilevel Linear and Quadratic Programs

V. VISWESWARAN\*, C. A. FLOUDAS\*\*

*Department of Chemical Engineering, Princeton University, Princeton, NJ 08544, USA*

M. G. IERAPETRITOU AND E. N. PISTIKOPOULOS

*Centre for Process Systems Engineering, Department of Chemical Engineering, Imperial College, London, SW7 2BY, U.K.*

**Abstract.** The paper presents a decomposition based global optimization approach to bilevel linear and quadratic programming problems. By replacing the inner problem by its corresponding KKT optimality conditions, the problem is transformed to a single yet non-convex, due to the complementarity condition, mathematical program. Based on the primal-dual global optimization approach of Floudas and Visweswaran (1990, 1993), the problem is decomposed into a series of primal and relaxed-dual subproblems whose solutions provide lower and upper bounds to the global optimum. By further exploiting the special structure of the bilevel problem, new properties are established which enable the efficient implementation of the proposed algorithm. Computational results are reported for both linear and quadratic example problems.

## 1. Introduction

Bilevel programming refers to optimization problems in which the constraint region is implicitly determined by another optimization problem, as follows:

$$\begin{aligned} & \min_x F(x, y) \\ & \text{s.t.} \\ & G(x, y) \leq 0 \end{aligned} \tag{P}$$

$$y \in \left\{ \begin{array}{l} \min_y f(x, y) \\ \text{s.t. } g(x, y) \leq 0 \\ x \in X, \quad y \in Y \end{array} \right\}$$

where  $G(x, y)$  is the vector valued function  $X \times Y \rightarrow R^p$ ,  $g(x, y)$  is the vector valued function  $X \times Y \rightarrow R^m$ , and  $X$  and  $Y$  are compact convex sets.

Problem (P) can be interpreted in the following way. At the higher level the decision maker (leader) has to choose first a vector  $x \in X$  to minimize his objective function  $F$ ; then in light of this decision the lower level decision maker (follower) has to select the decision vector  $y \in Y$  that minimizes his own objective  $f$ .

---

\* Current Address: Mobil Research and Development Corporation, Pennington, NJ 08534

\*\* Author to whom correspondence should be addressed

Applications of bilevel programming are diverse, including (i) design optimization problems of chemical plants where regions of different models should be examined (as for example in equilibrium calculations where the different regions correspond to different number and type of phases), (ii) long-range planning problems followed by short-term scheduling in chemical and other industries, (iii) hierarchical decision making policy problems in mixed economies, where policy makers at the top level influence the decisions of private individuals and companies, and (iv) energy consumption of private companies, which is affected by imported resources controlled by government policy.

Problem (P) has received a lot of attention, especially for the linear case. Broadly, one can distinguish two major classes of approaches for bilevel linear problems:

- (i) Enumeration techniques exploit the fact that an optimal solution to the bilevel problem is a basic feasible solution of the linear constraints involved at the lower and upper level and consequently must occur at an extreme point of the feasible set (e.g. the enumeration method by Candler and Townsley, 1982; “Kth Best algorithm” by Bialas and Karwan, 1984; B&B algorithm by Bard and Moore, 1990).
- (ii) Reformulation techniques based on the transformation of the original problem to a single optimization problem by employing the optimality KKT conditions of the lower level problem. For the solution of the resulting formulation the following algorithms have been developed: B&B techniques (Bard and Falk, 1982); mixed integer programming techniques (Fortuny and McCarl, 1981); parametric complementarity pivoting (Judice and Faustino, 1992); local optimization approaches for nonlinear programming such as penalty and barrier function methods (Anandalingam and White, 1990, White and Anandalingam, 1993) and global optimization techniques based on the reformulation of the complementarity slackness constraint to a separable quadratic reverse convex inequality constraint (Al-Khayyal et al. 1992) or the restatement of the original problem as a reverse convex program (Tuy et al. 1993, 1994).

For bilevel nonlinear problems, Bard (1983, 1984) developed a one-dimensional search algorithm that yields a locally optimal solution. However, it has been proven by Clark and Westerberg (1988), Ben-Ayed and Blair (1990) and Haurie et al. (1990) that the optimality conditions used by the previous author are not correct. Penalty function methods were used by Aiyoshi and Shimizu (1981), which do not guarantee the global optimal solution because of the non-convex nature of the problem.

In this paper a new algorithm of class (ii) is proposed for the case of convex outer level constraints ( $G(x, y)$ ) and linear inner level constraints ( $g(x, y)$ ). The approach takes full advantage of the special problem structure in order to employ the recently developed global optimization techniques based on primal-relaxed dual decomposition (Floudas and Visweswaran, 1990, 1993). The paper is organized as follows. In the next section, the bilevel linear problem is formulated followed by a brief discussion concerning the nature of the problem as well as the solution difficulties.

The new global optimization method is then presented in detail, a small example is used to demonstrate the main ideas and basic steps of the proposed approach and computational results for a battery of example problems are given. Finally, section 3 presents the extension of the proposed approach to linear-quadratic as well as quadratic-quadratic bilevel programming problems.

## 2. Bilevel linear programming problem

If all functions are linear, problem (P) gives rise to the following bilevel linear programming formulation:

$$\begin{aligned}
 & \min_x F(x, y) = c_1^T x + d_1^T y \\
 & \text{s.t.} \\
 & G(x, y) \leq 0 \\
 & y \in \left\{ \begin{array}{l} \min_y f(x, y) = c_2^T x + d_2^T y \\ \text{s.t. } g(x, y) = Ax + By - b \leq 0 \\ x \geq 0 \end{array} \right\}
 \end{aligned} \tag{P2}$$

For the sake of simplicity, the constraints  $G(x, y)$  will be ignored in the sequel. However, it is easy to show that the results obtained below hold in the presence of general convex constraints at the outer level. It should also be noted that any bounds on  $y$  are assumed to be incorporated into the inner level inequality constraints.

For the rest of the paper the following terminology will be used:

*Follower's Feasible region*

$$S(x) = \{y \mid g(x, y) \leq 0\}$$

*Follower's Rational Reaction Set*

$$RR(x) = \{y \in \operatorname{argmin} f(x, y) \mid y \in S(x)\}$$

*Inducible Region* which corresponds to the feasible region of problem (P)

$$IR = \{(x, y) \mid x \geq 0, y \in RR(x)\}$$

The solution of bilevel linear programming problems involves a number of interesting features:

- even for the linear case where the the follower's feasible region  $S$  is convex, the inducible region (IR) where the leader's objective should be minimized is a non-convex region. This is graphically illustrated in Figure 1, where the feasible region is shown as the shaded region (S), whereas the inducible region (IR) is the dashed non-convex region.

- Hansen *et al* (1990), proved that the bilevel linear programming problem is strongly NP-hard.
- The presence of dual degeneracy at the follower's problem, while not affecting the value of follower's objective function, can have an impact on the leader's objective. In this case the follower's choice among the multiple optima will be based on his willingness to cooperate with the leader. Two extreme cases concern the follower's acceptance of the leader's preferences regarding  $y$  (the tie cooperative case) and the follower's adopting the opposite of the leader's preferences (the tie non-cooperative case). The approach presented in this paper can handle both cases, but can be expected to be more efficient for the tie cooperative case.

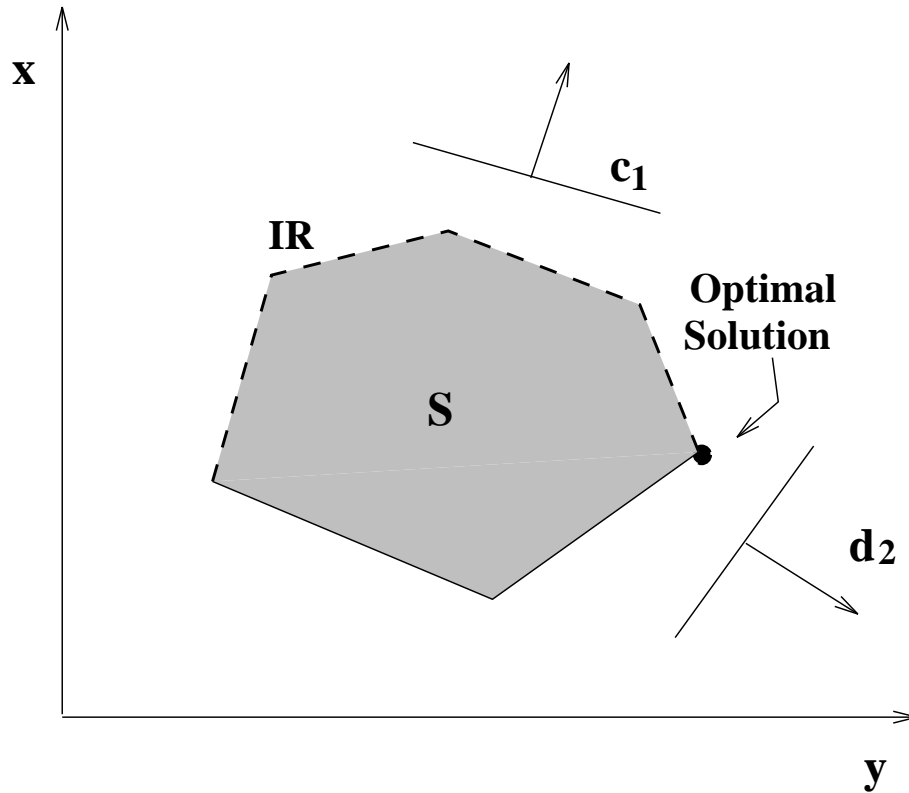


Figure 1. Non-convexity of bilevel linear problem

## 2.1. Equivalent Formulation

Rather than working with problem (P2) in its hierarchical form the analysis begins by converting it into a single mathematical program. This can be achieved by replacing the follower's optimization problem with the necessary and sufficient KKT optimality conditions. This results in the following problem:

$$\text{s.t.} \quad \left. \begin{array}{l} \min_{x,y,u} \quad c_1^T x + d_1^T y \\ d_2 + u^T B = 0 \\ u_i(Ax + By - b)_i = 0, \quad i = 1, \dots, m \\ Ax + By \leq b \\ x \geq 0, y \geq 0, u_i \geq 0, \quad i = 1, \dots, m \end{array} \right\} \quad (\text{P2S})$$

where  $u_i$  is the Lagrange multiplier of the  $i^{th}$  follower's constraint  $(Ax + By - b)_i$ ,  $i = 1, \dots, m$ . Note that the optimality conditions assume the existence of a stable point for the inner optimization problem, and therefore assume the satisfaction of an appropriate constraint qualification.

Problem (P2S) is a single nonlinear optimization problem, albeit non-convex due to the presence of bilinear terms in the complementarity conditions. Floudas and Visweswaran (1990, 1993) demonstrated that this class of problems can be solved to global optimality through their primal-dual decomposition algorithm (GOP) which transforms the original problem into a series of primal and relaxed-dual (RD) subproblems. The GOP algorithm was shown to have finite convergence to an  $\epsilon$ -global optimum. Here, by exploiting the special problem structure and introducing extra 0-1 variables to express the tightness of the follower's constraints a modified and more efficient algorithm is developed.

## 2.2. Mathematical Properties

Consider the following partition of the variables  $Y = u, X = (x, y)$  which satisfies Conditions (A) of the GOP algorithm (Floudas and Visweswaran, 1990, 1993). For fixed  $Y = Y^k$ , the primal problem can be written as:

$$\text{s.t.} \quad \left. \begin{array}{l} \min_{x,y} \quad c_1^T x + d_1^T y \\ Y_i^k(Ax + By - b)_i = 0, \quad i = 1, \dots, m \\ Ax + By \leq b \\ x \geq 0 \end{array} \right\} \quad (\text{P2S}')$$

Note that the KKT gradient conditions in problem (P2S), which are in the variables  $u$ , can be used directly in the dual problem. The solution to this primal problem, if feasible, yields the multipliers  $\lambda^k$  and  $\mu^k$  for the equality and inequality constraints in (P2S'). Note that when  $u_i^k = 0$ , the corresponding constraint drops out from the set of equality constraints, and there will be no multiplier for that constraint,

implying that  $\lambda_i^k = 0$  for this case. Conversely, when  $u_i^k > 0$ , the corresponding constraint is active, and therefore the value of  $\mu_i^k$  is zero.

The relaxed dual problem corresponding to (P2S') has the following form:

$$\left. \begin{array}{l} \min_{\mu_B, u} \quad \mu_B \\ \text{s.t.} \quad \mu_B \geq \min_{x, y} L(x, y, u, \mu^k, \lambda^k) \end{array} \right\} \text{ (RD)}$$

where  $L(x, y, u, \mu^k, \lambda^k)$  is the Lagrange function of the primal problem (P2S') and is given by

$$L(x, y, u, \mu^k, \lambda^k) = c_1^T x + d_1^T y + \sum_{i=1}^m (\mu_i^k + \lambda_i^k u_i) (Ax + By - b)_i \quad (1)$$

where  $(Ax + By - b)_i$  refers to the  $i^{th}$  inner constraint. Separating the terms in  $x$  and  $y$ , equation (1) can be rewritten as:

$$\begin{aligned} L(x, y, u, \mu^k, \lambda^k) = & [c_1^T + \sum_{i=1}^m (\mu_i^k + \lambda_i^k u_i) A_i] x + \\ & [d_1^T + \sum_{i=1}^m (\mu_i^k + \lambda_i^k u_i) B_i] y - \sum_{i=1}^m (\mu_i^k + \lambda_i^k u_i) b_i \end{aligned} \quad (2)$$

where  $A_i$  and  $B_i$  are the  $i^{th}$  rows of  $A$  and  $B$  respectively. The KKT gradient conditions of the primal problem (P2S'), for fixed Lagrange multipliers  $Y = Y^k = u^k$ , can be written as:

$$\nabla_x L(x, y, u^k, \mu^k, \lambda^k) = c_1 + \sum_{i=1}^m \mu_i^k A_i + \sum_{i=1}^m \lambda_i^k u_i^k A_i = 0 \quad (3)$$

$$\nabla_y L(x, y, u^k, \mu^k, \lambda^k) = d_1 + \sum_{i=1}^m \mu_i^k B_i + \sum_{i=1}^m \lambda_i^k u_i^k B_i = 0 \quad (4)$$

Incorporating (3) and (4) into (2) results in the following transformation:

$$L(x, y, u, \mu^k, \lambda^k) = \sum_{i=1}^m [\lambda_i^k (u_i - u_i^k) S_i - (\mu_i^k + \lambda_i^k u_i^k) b_i] \quad (5)$$

where  $S = Ax + By - b$  are slacks introduced for convenience. The advantage of the expression in (5) for the Lagrange function is that the following property can be established which effectively allows problem (RD) to be replaced by a sequence of subproblems corresponding to combinations of lower and upper bounds of the constraints.

**Property 2.1:** Suppose that the minimum value of the Lagrange function  $L^*$ ,  $L^*(\bar{x}, \bar{y}, u, \mu^k, \lambda^k) = \min_{x, y} L(x, y, u, \mu^k, \lambda^k)$  occurs at  $(\bar{x}, \bar{y})$ ; then,

$$L^*(\bar{x}, \bar{y}, u, \mu^k, \lambda^k) \geq \min_{B_j \in CB} \left\{ \begin{array}{l} \sum_{i=1}^m [\lambda_i^k (u_i - u_i^k) S_i^{B_j} - (\mu_i^k + \lambda_i^k u_i^k) b_i] \\ \lambda_i^k (u_i - u_i^k) \leq 0, \quad \forall S_i^{B_j} = S_i^L \\ \lambda_i^k (u_i - u_i^k) \geq 0, \quad \forall S_i^{B_j} = S_i^U \end{array} \right\}$$

where  $S = (Ax + By - b)$  are slacks ( $S \geq 0$ ) introduced for ease in the presentation;  $S_i^L, S_i^U$  are the lower and upper bounds on the constraints  $(Ax + By - b)_i$ , respectively;  $B_j$  corresponds to a combination of lower/upper bounds of constraints;  $S^{B_j}$  is the vector of lower/upper bounds of the constraints corresponding to the bound combination  $B_j$ ; and  $CB$  is the set of all bound combinations.

**Proof:** Consider the  $i^{th}$  component of the Lagrange function, corresponding to the  $i^{th}$  inner constraint. The minimum of this constraint corresponds to lies at a bound of  $(Ax + By - b)_i$ , the actual bound being determined by the sign of  $\lambda_i^k (u_i - u_i^k)$ . The following two cases can be distinguished:

(a) If  $u_i \geq u_i^k$  and  $\lambda_i^k \leq 0$ , or if  $u_i \leq u_i^k$  and  $\lambda_i^k \geq 0$ ,

$$\min_{x, y} L(x, y, u, \mu^k, \lambda^k) \geq \sum_{i=1}^m [\lambda_i^k (u_i - u_i^k) S_i^U - (\mu_i^k + \lambda_i^k u_i^k) b_i]$$

(b) If  $u_i \leq u_i^k$  and  $\lambda_i^k \geq 0$ , or if  $u_i \geq u_i^k$  and  $\lambda_i^k \leq 0$ ,

$$\min_{x, y} L(x, y, u, \mu^k, \lambda^k) \geq \sum_{i=1}^m [\lambda_i^k (u_i - u_i^k) S_i^L - (\mu_i^k + \lambda_i^k u_i^k) b_i]$$

From this, it follows that there exists a combination of bounds  $B_j$  of the constraints such that:

$$\min_{x, y} L(x, y, u, \mu^k, \lambda^k) \geq \sum_{i=1}^m [\lambda_i^k (u_i - u_i^k) S_i^{B_j} - (\mu_i^k + \lambda_i^k u_i^k) b_i] \quad \square$$

**Remark 2.1:** The above property preserves the important feature of the GOP algorithm that the solution of problem (RD) can be equivalently substituted by a series of optimization subproblems corresponding to different partitions of the Y-space.

**Remark 2.2:** It can be seen from equation (5) that the Lagrange function is essentially expressed in terms of the follower's constraints. This implies that from a

computational point of view, the complexity of the relaxed dual problem is determined by the number of active inner problem constraints (i.e. those constraints for which  $\lambda_i^k \neq 0$ ). This can be of great significance in problems with large number of variables but few constraints. For instance, for the case of two  $x$  and two  $y$  variables with two constraints, the number of subproblems that would be needed is reduced from  $2^4$  to only 3 (since the combination of the zero upper bounds for all the constraints results in redundant RD subproblem).

### 2.3. Introduction of 0-1 variables

It is clear that each combination of the  $u$  variables corresponds to a vertex of the followers feasible region. However, different combinations with the same set of nonzero  $u_i$  correspond to the same vertex. It is desirable to avoid such nonzero combinations from being generated more than once. This can be ensured by the introduction of binary variables, as shown below.

Consider the set of binary variables  $a_i$ ,  $i = 1, \dots, m$ , associated with each one of the follower's constraints as follows:

$$a_i = \begin{cases} 1, & \text{if constraint } (Ax + By - b)_i \text{ is active} \\ 0, & \text{otherwise} \end{cases}$$

The following set of *big-M* constraints are also introduced to establish one-to-one correspondence between the multiplier  $u_i$  of constraint  $i$  and the corresponding 0-1 variable  $a_i$ :

$$(1/M)a_i \leq u_i \leq Ma_i \quad (6)$$

Constraint (6) implies that if  $a_i = 0 \Rightarrow 0 \leq u_i \leq 0 \Rightarrow u_i = 0$ , i.e. the multiplier is also zero, forcing the corresponding constraint to be tight, whereas if  $a_i = 1 \Rightarrow (1/M) \leq u_i \leq M$ , the associated multiplier has nonzero value implying an inactive constraint.

The incorporation of constraints (6) along with the 0-1 variables  $a_i$  into problem (P2S) results in:

$$\left. \begin{array}{l} \min_{x,y,u} \quad c_1^T x + d_1^T y \\ \text{s.t.} \quad d_2 + u^T B = 0 \\ a_i (Ax + By - b)_i = 0, \quad i = 1, \dots, m \\ u_i \leq Ma_i, \quad i = 1, \dots, m \\ a_i \leq Mu_i, \quad i = 1, \dots, m \\ Ax + By \leq b \\ x \geq 0, y \geq 0, u \geq 0, a_i = \{0-1\} \end{array} \right\} \quad (\text{P3S})$$

By augmenting the  $Y$ -vector to include the 0-1 variables, the following primal problem can be derived for  $Y = Y^k = (u^k, a^k)$ :



$$\text{s.t. } \left. \begin{array}{l} \min_{x,y,u} \quad c_1^T x + d_1^T y \\ a_i^k (Ax + By - b)_i = 0, \quad i = 1, \dots, m \\ Ax + By \leq b \\ x \geq 0, y \geq 0 \end{array} \right\} \quad (\text{P3S}')$$

with a corresponding relaxed dual problem of the following form:

$$\text{s.t. } \left. \begin{array}{l} \min_{\mu_B, u, a} \quad \mu_B \\ \mu_B \geq \min_{x,y} \quad L(x, y, a, \mu^k, \lambda^k) \\ L(x, y, a, \mu^k, \lambda^k) = c_1^T x + d_1^T y + \sum_{i=1}^m (\mu_i^k + \lambda_i^k a_i) (Ax + By - b)_i \\ u_i \leq M a_i, \quad i = 1, \dots, m \\ a_i \leq M u_i, \quad i = 1, \dots, m \end{array} \right\} \quad (\text{RD}')$$

Using the KKT optimality conditions for the primal problem (P3S') the Lagrange function can be written as

$$L(x, y, a, \mu^k, \lambda^k) = \sum_{i=1}^m [\lambda_i^k (a_i - a_i^k) (Ax + By - b)_i - (\mu_i^k + \lambda_i^k a_i^k) b_i] \quad (7)$$

Property 2.1 can then be recast as follows:

$$L^*(\bar{x}, \bar{y}, u, \mu^k, \lambda^k) \geq \min_{B_j \in CB} \left\{ \begin{array}{l} \sum_{i=1}^m [\lambda_i^k (a_i - a_i^k) S_i^{B_j} - (\mu_i^k + \lambda_i^k a_i^k) b_i] \\ \lambda_i^k (a_i - a_i^k) \leq 0, \quad \forall S_i^{B_j} = S_i^L \\ \lambda_i^k (a_i - a_i^k) \geq 0, \quad \forall S_i^{B_j} = S_i^U \end{array} \right\}$$

Consider the  $i^{th}$  term. It is clear that if  $a_i^k = 0$ , the corresponding constraint would have been absent from the primal problem (P3S'), leading to  $\lambda_i^k = 0$ , so that this term would be absent from the summation. Therefore, only the case of  $a_i^k = 1$  is important. Then, since  $a_i$  is always less than or equal to  $a_i^k$ , the minimum of  $L(x, y, a, \mu^k, \lambda^k)$  occurs at the lower (upper) bound of  $(Ax + By - b)_i$  if  $\lambda_i^k \leq 0$  ( $\lambda_i^k \geq 0$ ). Therefore, it is sufficient to set each active constraint in the summation to the appropriate bound, and the following result is always true:

Only one relaxed dual problem is solved at every iteration regardless of the size of the problem.

**Remark 2.4:** Another advantage of (PS3) problem formulation is that additional constraints (integer cuts) in the 0-1 variables,  $a_i$ , can be used together with the Lagrangian cut to improve the solution efficiency of the resulting MILP relaxed dual problem. In particular, as has been showed by Hansen et al. (1990), in any optimal solution of bilevel programming problem (PS1) the active constraints of the follower's problem satisfy the following conditions:

$$\begin{aligned} \sum_{i \in I_p(i)} a_i &\geq 1, \text{ if } d_i > 0, i = 1, \dots, m \\ \sum_{i \in I_n(i)} a_i &\geq 1, \text{ if } d_i < 0, i = 1, \dots, m \end{aligned}$$

where  $I_p(i), I_n(i)$  are the sets of constraints in which variable  $y_i$  appears with positive and negative sign, respectively. Also, an active set strategy suggests that:

$$\sum_{i=1}^m a_i \leq |y|$$

where  $|y|$  is the cardinality of the follower's decision vector  $y$ . It can be seen that these and other preprocessing steps can be done on the binary variables to eliminate certain combinations.

**Remark 2.5:** In cases where the primal problem is infeasible, the following relaxed primal problem is formulated and solved:

$$\left. \begin{aligned} \min_{x, y, s_I, s_A^+, s_A^-} \quad & \theta = \sum_{i=1}^{m_A} (s_{iA}^+ + s_{iA}^-) + \sum_{i=1}^{m_I} s_{iI} \\ \text{s.t.} \quad & a^k \top (A_A x + B_A y - b_A) + s_A^+ - s_A^- = 0 \\ & A_I x + B_I y - b_I - s_I \leq 0 \\ & x, y, s_A^+, s_A^-, s_I \geq 0 \end{aligned} \right\} \quad (\text{P4I})$$

where  $m_A, m_I$  are the number of active and inactive constraints at the current iteration, respectively;  $s_I, s_A^+, s_A^-$  are slacks variables that are introduced in order to minimize the sum of infeasibilities. The Lagrange function of problem (P4I) is:

$$\begin{aligned} L(x, y, s_I, s_A^+, s_A^-) = \theta + \sum_{i=1}^m \{ & \lambda_i^k [a_i (Ax + By - b) + s_{iA}^+ - s_{iA}^-] \\ & + \mu_i^k [Ax + By - b - s_{iI}]_i \} \end{aligned} \quad (8)$$

which can be transformed by using the optimality conditions to the following form:

$$L(x, y, ) = \sum_{i=1}^m \lambda_i^k (a_i - a_i^k) (Ax + By - b)_i + \bar{\theta} \quad (9)$$

where  $\bar{\theta}$  is the optimal solution of problem (P4I). Based on this, the following feasibility cut can be introduced in problem (RD):

$$\sum_{i=1}^m \lambda_i^k (a_i - a_i^k) (Ax + By - b)_i + \bar{\theta} \leq 0 \quad (10)$$

## 2.4. Modified GOP Algorithm

Based on the above analysis, a modified algorithm for global optimization of bilevel linear programming programs is now described in the following steps:

### Step 0: Initialization of Parameters.

Define the storage parameters  $\mu_B^{stor}(K^{max})$ ,  $y^{stor}(K^{max})$  and  $y^k(K^{max})$  over the maximum expected number of iterations  $K^{max}$ . Define  $P^{UBD}$ ,  $R^{LBD}$  as the upper and lower bounds obtained from the primal and relaxed dual problems, respectively. Set  $\mu_B^{stor}(K^{max}) = U$ ,  $P^{UBD} = U$ ,  $R^{LBD} = L$ , where U and L are very large positive and negative numbers, respectively. Select a starting point  $Y^1$ , set the iteration counter equal to 1, and select a tolerance for convergence  $\epsilon$ . Find lower and upper bounds on the inner constraints by solving the following problems:

$$\left. \begin{array}{ll} \min_{x,y} & \pm (Ax + By - b)_i \\ \text{s.t.} & (Ax + By - b)_j \leq 0 \quad j = 1, \dots, m, \quad j \neq i \\ & x, y \geq 0 \end{array} \right\} (PB^i) \quad i = 1, \dots, m$$

### Step 1: Primal Problem.

Store the value  $Y^k$ . Solve problem (P4S), store the Lagrange multipliers  $\lambda_A^k$  and update the upper bound so that  $P^{UBD} = \min\{P^{UBD}, Z^k\}$ , where  $Z^k$  is the current primal objective. If the primal is infeasible a relaxed primal is solved and the Lagrange multipliers are stored.

### Step 2: Relaxed-Dual Problem.

Formulate the Lagrange function corresponding to the current primal problem as described in equation (7) and add this as a constraint to the relaxed-dual problem. For iterations when the primal problem is infeasible use a cut of the form (10) in the constraints corresponding to that iteration. Solve the resulting (RD) and store the solution  $\mu_B^{stor}(k)$ ,  $y^{stor}(k)$ .

### Step 3: Selecting a new lower bound $R^{LBD}$ .

From the stored values  $\mu_B^{stor}$  select the minimum  $\mu_B^{min}$  and set  $R^{LBD} = \mu_B^{min}$ ,  $Y^{k+1} = Y^{min}$  the corresponding stored value of Y. Delete  $\mu_B^{min}$ ,  $Y^{min}$  from the stored set.

### Step 4: Check for convergence.

Check if  $R^{LBD} \geq P^{UBD} - \epsilon$ . If yes, stop. Else set  $k=k+1$  and return to Step 1.

## 2.5. An Illustrating Example

Consider the following two-level linear program (from Bard, 1983):

$$\begin{aligned}
& \min_x F(x, y) = x + y \\
& \text{s.t.} \quad -x \leq 0 \\
& \quad \min_y f(x, y) = -5x - y \\
& \quad \text{s.t.} \quad -x - 0.5y \leq -2 \\
& \quad \quad -0.25x + y \leq 2 \\
& \quad \quad x + 0.5y \leq 8 \\
& \quad \quad x - 2y \leq 2 \\
& \quad \quad -y \leq 0
\end{aligned}$$

Considering the KKT optimality conditions of the inner problem and introducing the binary variables  $a_i$ , the bilevel problem becomes:

$$\begin{aligned}
& \min_{x, y, u} F(x, y) = x + y \\
& \text{subject to:} \\
& \left. \begin{aligned}
& -x \leq 0 \\
& -x - 0.5y \leq -2 \\
& -0.25x + y \leq 2 \\
& x + 0.5y \leq 8 \\
& x - 2y \leq 2 \\
& -y \leq 0
\end{aligned} \right\} \begin{array}{l} \text{Feasibility} \\ \text{Constraints} \end{array} \\
& \left. \begin{aligned}
& a_1(-x - 0.5y + 2) = 0 \\
& a_2(-0.25x + y - 2) = 0 \\
& a_3(x + 0.5y - 8) = 0 \\
& a_4(x - 2y - 2) = 0 \\
& a_5(-y) = 0
\end{aligned} \right\} \begin{array}{l} \text{Complementarity} \\ \text{Constraints} \end{array} \\
& -0.5u_1 + u_2 + 0.5u_3 - 2u_4 - u_5 = 1 \} \begin{array}{l} \text{Stationarity} \\ \text{Constraint} \end{array} \\
& \left. \begin{aligned}
& u_i \leq M a_i, \quad i = 1, \dots, 5 \\
& a_i \leq M u_i, \quad i = 1, \dots, 5
\end{aligned} \right\} \begin{array}{l} \text{Logical} \\ \text{Constraints} \end{array} \\
& a_i \in \{0, 1\}, \quad u_i \geq 0
\end{aligned}$$

Before starting the iterations, problem  $(PB^i)$  is formulated and solved for each constraint  $i=1, \dots, 5$  which yields the following lower bounds for the constraints:

$$LB_1 = -6, \quad LB_2 = -3, \quad LB_3 = -6, \quad LB_4 = -8, \quad LB_5 = -4$$

**Iteration 1:** Consider a starting point of  $a_i = 1, i = 1, \dots, 5$  which corresponds to all inner constraints being active. The primal problem can then be written as:

$$\begin{aligned}
& \min_{x,y} F(x,y) = x + y \\
& \text{subject to:} \\
& \quad -x \leq 0 \\
& \quad -x - 0.5y = -2 \\
& \quad -0.25x + y = 2 \\
& \quad x + 0.5y = 8 \\
& \quad x - 2y = 2 \\
& \quad -y = 0
\end{aligned}$$

This problem is infeasible; therefore, the following relaxed primal problem (P4I) is formulated and solved:

$$\begin{aligned}
& \min_{x,y,s_i^+,s_i^-} \sum_{i=1}^5 (s_i^+ + s_i^-) \\
& \text{subject to:} \\
& \quad -x \leq 0 \\
& \quad -x - 0.5y + s_1^+ - s_1^- = -2 \\
& \quad -0.25x + y + s_2^+ - s_2^- = 2 \\
& \quad x + 0.5y + s_3^+ - s_3^- = 8 \\
& \quad x - 2y + s_4^+ - s_4^- = 2 \\
& \quad -y + s_5^+ - s_5^- = 0
\end{aligned}$$

The Lagrange cut (14) formulated for this problem has the following form:

$$6a_1 + 3a_2 + 6a_3 + 2a_4 + 2a_5 - 10 \leq 0$$

The (RD) subproblem is shown below:

$$\begin{aligned}
& \min_{a,u} \mu_B \\
& \text{subject to:} \\
& \quad 6a_1 + 3a_2 + 6a_3 + 2a_4 + 2a_5 - 10 \leq 0 \\
& \quad -0.5u_1 + u_2 + 0.5u_3 - 2u_4 - u_5 = 1 \\
& \quad u_i \leq Ma_i, \quad i = 1, \dots, 5 \\
& \quad a_i \leq Mu_i, \quad i = 1, \dots, 5 \\
& \quad a_2 + a_3 \geq 1 \\
& \quad \sum_{i=1}^5 a_i \leq 1 \\
& \quad \mu_B \leq U
\end{aligned}$$

where  $U$  is a very large positive number. The solution of this problem is  $\mu_B = U$  and  $a_3 = 1$  the third constraint active.

**Iteration 2:** For  $a_3 = 1$ , the primal problem is:

$$\begin{aligned} \min_{x,y} \quad & F(x,y) = x + y \\ \text{subject to:} \quad & -x \leq 0 \\ & -x - 0.5y \leq -2 \\ & -0.25x + y \leq 2 \\ & x + 0.5y = 8 \\ & x - 2y \leq 2 \\ & -y \leq 0 \end{aligned}$$

The primal is feasible and its solution yields:

$$x = 7.2, \quad y = 1.6, \quad Z^2 = 8.8$$

The Lagrange function formulated from the second primal problem is:

$$L(a) = 1.6 + 7.2a_3$$

and the relaxed dual subproblem:

$$\begin{aligned} \min_{a,u} \quad & \mu_B \\ \text{subject to:} \quad & \mu_B \geq 1.6 + 7.2a_3 \\ & 6a_1 + 3a_2 + 6a_3 + 2a_4 + 2a_5 - 10 \leq 0 \\ & -0.5u_1 + u_2 + 0.5u_3 - 2u_4 - u_5 = 1 \\ & u_i \leq Ma_i, \quad i = 1, \dots, 5 \\ & a_i \leq Mu_i, \quad i = 1, \dots, 5 \\ & a_2 + a_3 \geq 1 \\ & \sum_{i=1}^5 a_i \leq 1 \\ & \mu_B \leq 8.8 \\ & \mu_B \geq L \end{aligned}$$

Its solution is:

$$\mu_B = 1.6, \quad a_2 = 1$$

Thus, after the second iteration,  $P^{UBD} = 8.8$ ,  $R^{LBD} = 1.6$  and  $a_2 = 1$ .

**Iteration 3:** The primal problem is solved for  $a_2 = 1$ . The solution of this problem gives:

$$x = 0.889, \quad y = 2.222, \quad Z^3 = 3.111$$

The Lagrange function from the third primal problem is:

$$L(a) = 1.777 + 1.333a_2$$

and the relaxed dual subproblem:

$$\begin{aligned} & \min_{a,u} \quad \mu_B \\ & \text{subject to:} \\ & \mu_B \geq 1.777 + 1.333a_2 \\ & \mu_B \geq 1.6 + 7.2a_3 \\ & 6a_1 + 3a_2 + 6a_3 + 2a_4 + 2a_5 - 10 \leq 0 \\ & -0.5u_1 + u_2 + 0.5u_3 - 2u_4 - u_5 = 1 \\ & u_i \leq Ma_i, \quad i = 1, \dots, 5 \\ & a_i \leq Mu_i, \quad i = 1, \dots, 5 \\ & a_2 + a_3 \geq 1 \\ & \sum_{i=1}^5 a_i \leq 1 \\ & \mu_B \leq 3.111 \\ & \mu_B \geq 1.6 \end{aligned}$$

Its solution is:

$$\mu_B = 3.111, \quad a_2 = 1$$

Thus, after the second iteration,  $P^{UBD} = 1.6, R^{LBD} = 1.6, a_2 = 1$  and  $(x,y) = (0.889, 2.222)$  which corresponds to the global optimum.

The KKT optimality conditions of the inner problem imply that either the second or the third constraint has to be active leading to the hatched IR shown in Figure 2. From all these points (feasible region of the RD subproblem) the solution of the primal problem yields the one that minimize the leader's objective function (point A in Figure 2).

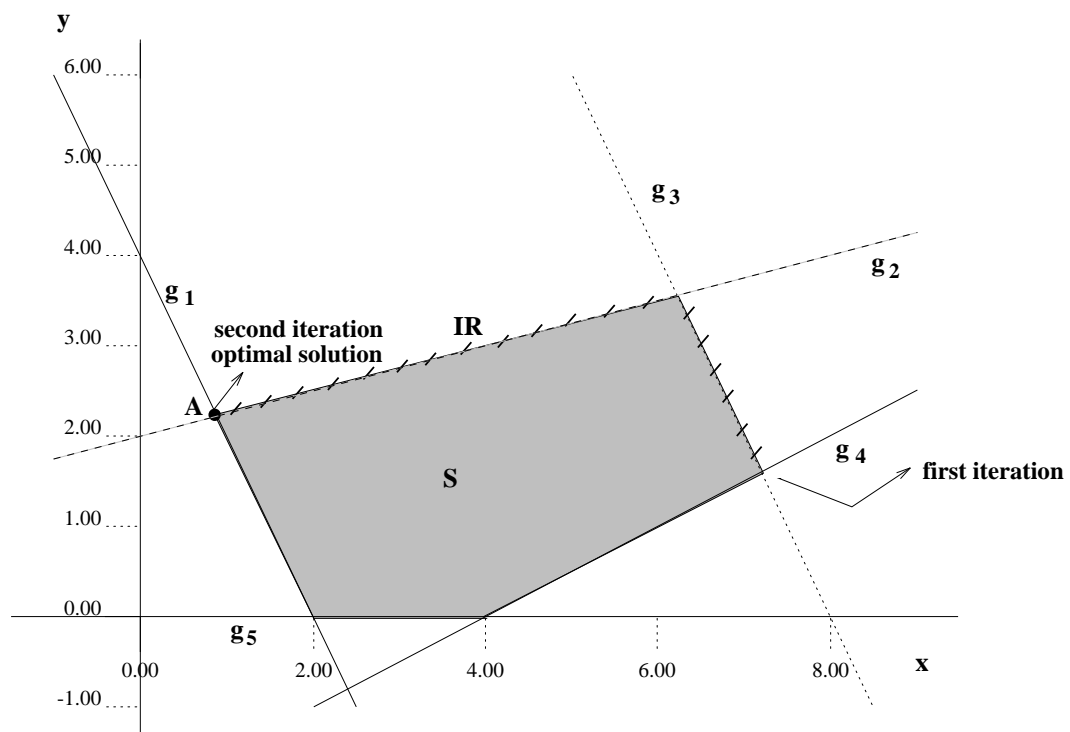


Figure 2. Feasible Region of Illustrating Example



## 2.6. Computational Experience

The modified GOP algorithm has been coded in C language and tested for a series of small example problems appeared in the literature. The results are summarized in Table 1.

Table 1. Results for small linear examples

EXAMPLES	$n_x$	$n_y$	Outer Constraints	Inner Constraints	Iterations	CPU time (s)
EX1	2	3	2	6	5	0.59
EX2	1	1	1	7	2	0.11
EX3	1	1	1	6	3	0.29
EX4	6	3	6	10	3	0.75
EX5	1	1	1	5	3	0.29
EX6	1	2	1	4	2	0.16
EX7	1	1	1	4	3	0.23
EX8	1	1	1	4	3	0.22
EX9	1	1	1	5	3	0.29
EX10	1	2	2	4	2	0.16
EX11	2	3	3	6	5	0.82

A number of randomly generated problems with the same characteristics as in Bard and Moore (1990), and Hansen et al. (1990) with 40% and 33% density, have also been considered. The results are summarized in Table 2. All computations are performed using CPLEX for the solution of linear problems using a HP-730. Performance measures include CPU time and the number of iterations required to obtain the global solution. The main parameters considered are the number of the follower's constraints and the numbers  $n_x, n_y$  of the leader's and the follower's variables. Problems involving 12-17 constraints and 40-50 variables were solved.

As expected the CPU time and the number of iterations increases with the size of the problem. Notice however, that the number of required iterations remains relatively low. Also, as seen in Table 2, large differences in computation effort are observed between problems of the same size (for example, for the case of 26 outer and 14 inner variables, computation times of 867 and 81 sec have been reported for two different examples).

## 3. Linear-quadratic and quadratic-quadratic bilevel problems

In this section the solution approach is extended to consider the linear/quadratic as well as the quadratic/quadratic bilevel programming problems of the following

Table 2. Computational Results for randomly generated linear problems

DENSITY	$n_x$	$n_y$	Inner Constraints	Iterations	CPU time (s)
40%	28	12	12	7	8.4
40%	28	12	12	57	121.0
40%	28	12	12	9	8.0
40%	28	12	12	14	6.8
40%	26	14	14	26	81.1
40%	26	14	14	88	867.6
40%	25	15	15	12	42.8
40%	25	15	15	20	52.1
40%	24	17	17	5	34.0
40%	30	15	15	36	129.8
40%	35	15	15	26	282.3
40%	35	15	15	85	455.3
33%	27	13	13	54	249.6
33%	27	13	13	17	52.6
33%	27	13	13	21	98.8

general form:

$$\begin{aligned}
 & \min_x F(x, y) \\
 & s.t. \\
 & y \in \left\{ \begin{array}{l} \min_y f(x, y) \\ s.t. \quad Ax + By \leq b \\ x \geq 0 \end{array} \right\} \quad (P')
 \end{aligned}$$

where  $F(x, y)$  is a convex function of  $x$  and  $y$ , and  $f(x, y) = d_2 y + x^\top Q_1^2 y + y^\top Q_2^2 y$ . For sake of simplicity, it is assumed that  $F(x, y) = c_1^\top x + d_1^\top y$ . It can easily be shown, however, that the following analysis is valid for any convex form of  $F(x, y)$ . It is also assumed that  $f(x, y)$  is a convex quadratic function. Then, the KKT conditions for the inner problem are both necessary and sufficient for inner optimality which preserves the equivalence of problems (P') and (PS') below:

$$\left. \begin{aligned}
 & \min_{x, y, u} F(x, y) = c_1^\top x + d_1^\top y \\
 & s.t. \quad Ax + By \leq b \\
 & \quad 2y^\top Q_2^2 + x^\top Q_1^2 + u^\top B_2 + d_2 = 0 \\
 & \quad u_i (Ax + By - b)_i = 0, \quad i = 1, \dots, m \\
 & \quad x \in X, y \in Y, u \geq 0
 \end{aligned} \right\} (PS')$$

Introducing the set of 0-1 variables  $a_i$  results in following equivalent formulation:

$$\left. \begin{array}{l} \min_{x,y,a,u} F(x,y) = c_1^T x + d_1^T y \\ \text{s.t. } Ax + By \leq b \\ 2y^T Q_2^2 + x^T Q_1^2 + u^T B_2 + d_2 = 0 \\ a_i(Ax + By - b)_i = 0, \quad i = 1, \dots, m \\ u_i \leq M a_i, \quad i = 1, \dots, m \\ a_i \leq M u_i, \quad i = 1, \dots, m \\ x \in X, y \in Y, a_i = \{0-1\}, u_i \geq 0, i = 1, \dots, m \end{array} \right\} \quad (\text{P3S}')$$

As in the linear case, the variables can be partitioned into  $Y = (a, u)$ , and  $X = (x, y)$ . Then, for fixed  $Y = Y^k$  the primal problem becomes

$$\left. \begin{array}{l} \min_{x,y} F(x,y) = c_1^T x + d_1^T y \\ \text{s.t. } Ax + By \leq b \\ 2y^T Q_2^2 + x^T Q_1^2 + u^k B_2 + d_2 = 0 \\ a_i^k(Ax + By - b)_i = 0, \quad i = 1, \dots, m \\ x \in X, y \in Y \end{array} \right\} \quad (\text{P4S}')$$

The Lagrange function of problem (P4S') is:

$$\begin{aligned} L(x, y, a, u, \mu^k, \lambda^k) &= c_1^T x + d_1^T y + \sum_{i=1}^m [(\mu_i^k + \lambda_i^k a_i) \cdot (Ax + By - b)_i] \\ &\quad + \sum_{i=1}^{n_y} \nu_i^k [2y^T Q_2^2 + x^T Q_1^2 + u^T B_2 + d_2]_i \end{aligned}$$

Separating the terms in  $x$  and  $y$ , this can be rewritten as:

$$\begin{aligned} L(x, y, a, u, \mu^k, \lambda^k) &= [c_1 + \sum_{i=1}^m (\mu_i^k + \lambda_i^k u_i) A_i + \sum_{i=1}^{n_y} \nu_i^k Q_{1i}^2]^T x \\ &\quad + [d_1 + \sum_{i=1}^m (\mu_i^k + \lambda_i^k u_i) B_i + \sum_{i=1}^{n_y} 2\nu_i^k Q_{2i}^2]^T y \\ &\quad - \sum_{i=1}^m (\mu_i^k + \lambda_i^k u_i) b_i + \sum_{i=1}^{n_y} \nu_i^k d_i^2 \end{aligned}$$

Using the KKT gradient conditions for problem (P4S'), the Lagrange function can be reduced to

$$\begin{aligned} L(x, y, a, u, \mu^k, \lambda^k, \nu^k) &= \sum_{i=1}^m \lambda_i^k (a_i - a_i^k) (Ax + By - b)_i \\ &\quad + \sum_{i=1}^{n_y} \nu_i^k (u_i - u_i^k) B_i + (c_1^T x + d_1^T y)^k \end{aligned}$$

Then, it is obvious that Property 2.1 holds:

$$\begin{aligned} \min_{x,y} L(x, y, a, u, \mu^k, \lambda^k, \nu^k) \geq & \sum_{i=1}^m \lambda_i^k (a_i - 1) (Ax + By - b)_i^L \\ & + \sum_{i=1}^{n_y} \nu_i^k (u_i - u_i^k) B_i + (c_1^T x + d_1^T y)^k \end{aligned}$$

and consequently only one relaxed dual subproblem has to be solved per iteration.

**Remark 3.1** Since the stationary conditions are functions of  $X$  and  $Y$  variables, they appear to both primal and relaxed-dual subproblems. Moreover, for the case of quadratic outer objective  $F(x, y)$  the primal problem corresponds to a nonlinear programming problem. However, under the convexity assumptions it can be solved using a conventional NLP solver.

### 3.1. Computational Studies

In this section, four examples are presented to highlight the performance of the proposed approach compared to the existing ones.

#### *Example 1*

This example is taken from Bard (1988).

$$\begin{aligned} \min_x \quad & F(x, y) = (x - 5)^2 + (2y + 1)^2 \\ \text{s.t.} \quad & -x \leq 0 \\ & \min_y \quad f(x, y) = (y - 1)^2 - 1.5xy \\ \text{s.t.} \quad & -3x + y \leq -3 \\ & x - 0.5y \leq 4 \\ & x + y \leq 7 \\ & -y \leq 0 \end{aligned}$$

Non-convexity implies the existence of local optima at (1,0) and (5,2). Global optimization approach requires 8 iterations to reach the global minimum (1,0);  $F = 17$ ,  $f = 1$ , within 3.3 CPU s in Sparc10 using GAMS/MINOS for the solution of nonlinear primal subproblem and GAMS/SCICONIC for mixed integer relaxed-dual subproblem.

#### *Example 2*

This example is taken from Shimizu and Aiyoshi (1981).

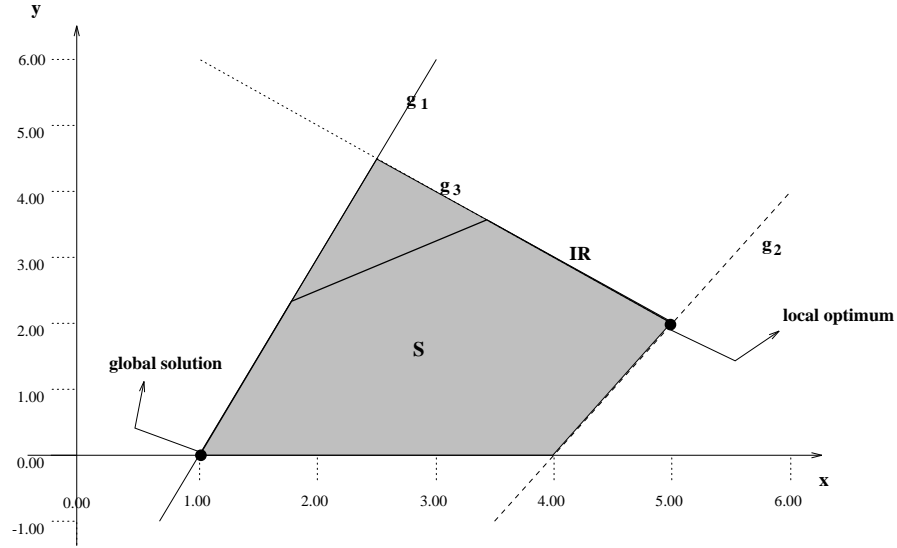


Figure 3. Non-convexity of bilevel quadratic problem

$$\begin{aligned}
 & \min_x F(x, y) = x^2 + (y - 10)^2 \\
 & \text{s.t. } x \leq 15 \\
 & \quad -x + y \leq 0 \\
 & \quad -x \leq 0 \\
 & \min_y f(x, y) = (x + 2y - 30)^2 \\
 & \text{s.t. } x + y \leq 20 \\
 & \quad 0 \leq y \leq 20
 \end{aligned}$$

The modified GOP algorithm identified the global solution  $(x, y) = (10, 10)$ ;  $F = 1000$ ,  $f = 0$ , in only two iterations within 0.12 CPU s in HP-730 using GAMS/MINOS for the solution of nonlinear primal and GAMS/CPLEX for the MILP relaxed-dual subproblems.

### Example 3

This example is also from Shimizu and Aiyoshi (1981).

$$\begin{aligned}
 & \min_x F(x, y) = (x_1 - 30)^2 + (x_2 - 20)^2 - 20y_1 + 20y_2 \\
 & \text{s.t. } x_1 + 2x_2 \geq 30 \\
 & \quad x_1 + x_2 \leq 25 \\
 & \quad x_2 \leq 15 \\
 & \min_y f(x, y) = (x_1 - y_1)^2 + (x_2 - y_2)^2 \\
 & \text{s.t. } 0 \leq y_1 \leq 10 \\
 & \quad 0 \leq y_2 \leq 10
 \end{aligned}$$

For this example also two iterations were required for the GOP to find the global solution  $(x_1, x_2, y_1, y_2) = (20, 5, 10, 5)$ ;  $F = 225$ ,  $f = 100$ , in 0.12 CPU s.

For both the previous examples, the reported penalty method of Shimizu and Aiyoshi (1981) converges to near optimal solution as the penalty coefficient ( $r$ ) decreases. Also, ill-conditioning problems were observed for small values of  $r$ .

#### Example 4

This example is taken from Aiyoshi and Shimizu (1984).

$$\begin{aligned}
 & \min_x \quad F(x, y) = 2x_1 + 2x_2 - 3y_1 - 3y_2 - 60 \\
 & \text{s.t.} \quad x_1 + x_2 + y_1 - 2y_2 - 40 \leq 0 \\
 & \quad \quad 0 \leq x_1 \leq 50 \\
 & \quad \quad 0 \leq x_2 \leq 50 \\
 & \quad \min_y \quad f(x, y) = (y_1 - x_1 + 20)^2 + (y_2 - x_2 + 20)^2 \\
 & \text{s.t.} \quad -x_1 + 2y_1 \leq -10 \\
 & \quad \quad -x_2 + 2y_2 \leq -10 \\
 & \quad \quad -10 \leq y_1 \leq 20 \\
 & \quad \quad -10 \leq y_2 \leq 20
 \end{aligned}$$

The implementation of the modified GOP algorithm for this problem leads to the global minimum solution  $(x_1, x_2, y_1, y_2) = (0, 0, -10, -10)$ ,  $F = 0$ ,  $f = 200$ , after 8 iterations within 0.8 CPU s using GAMS/MINOS for the solution of LP primal and GAMS/CPLEX for the MILP relaxed-dual subproblems. Notice that the penalty solution method of Aiyoshi and Shimizu fails to identify the global optimum, converging to the local minimum  $(x_1, x_2, y_1, y_2) = (25, 30, 5, 10)$   $F = 5$ ,  $f = 100$ .

## 4. Conclusions

This paper presents a global optimization approach to bilevel linear and quadratic programming problems. The problem is decomposed into a series of primal and relaxed dual problems whose solutions provide lower and upper bounds to the global optimum. The exploitation of the special structure of the problem results in simplified primal and relaxed dual subproblems. In particular, the primal problem is a single linear/convex optimization problem (corresponding to an active set of the follower's constraints) whereas the relaxed dual problem is solved through **one** single subproblem. The simplified primal and relaxed dual problems result in a modified algorithm that is computationally efficient. The application of the modified algorithm is shown through an illustrating linear example that details the steps of the proposed approach. The results on a set of randomly generated examples highlight the efficiency of the new algorithm.

## Acknowledgments

Financial support from the Commission of European Communities under grant ERB CHBI CT93 0484 is gratefully acknowledged. E. N. Pistikopoulos and C. A. Floudas are also grateful to NATO for partial financial support of this work under a collaborating research grant CRG 940617. Visweswaran and Floudas acknowledge financial support from the National Science Foundation under grant CTS-9221411.

## References

1. Anandalingam G. and D.J. White (1990). A solution method for the Linear Static Stackelberg Problem Using Penalty Method. *IEEE Trans. Aut. Control*, **35**, 1170.
2. Aiyoshi, E and K. Shimizu (1981). Hierarchical Decentralized systems and its new solution by a Barrier Method. *IEEE Trans. Systems, Man and Cybernetics*, **SMC-11**, 444.
3. Aiyoshi, E and K. Shimizu (1984). A solution method for the Static Constraint Stackelberg Problem Via Penalty Method. *IEEE Trans. Aut. Control*, **AC-29**, 1111.
4. Al-Khayyal F., R. Horst and P. Pardalos (1992). Global Optimization on Concave Functions Subject to Quadratic Constraints: An Application in nonlinear Bilevel Programming. *Annals Oper. Res.*, **34**, 125.
5. Bard, J.F. (1983). An Algorithm for Solving the General Bilevel Programming Problem. *Math. Oper. Res.*, **8**, 260.
6. Bard, J.F. (1984). Optimality Conditions for the Bilevel Programming Problem. *Nav. Res. Log. Q.*, **31**, 13.
7. Bard, J.F. (1988). Convex Two-Level Optimization. *Math. Prog.*, **40**, 15.
8. Bard, J.F. and J.E. Falk (1982). An Explicit Solution to the Multi-Level programming Problem. *Comp. Oper. Res.*, **9**, 77.
9. Bard, J.F. and J.T. Moore (1990). A Branch and Bound Algorithm for the Bilevel Programming Problem. *SIAM J. Scien. Stat. Comp.*, **11**, 281.
10. Ben-Ayed O. and C. E. Blair (1989). Computational Difficulties of Bilevel Linear Programming. *Oper. Res. Tech. Notes*, 556.
11. Bialas, W.F. and M.H. Karwan (1984). Two-Level Linear Programming. *Mang. Sci.*, **30**, 1004.
12. Candler, W. and R. Townsley (1982). A Linear Two-Level Programming Problem. *Comp. Oper. Res.*, **9**, 59.
13. Clark, P.A. and A.W. Westerberg (1988). A Note on the Optimality Conditions for the Bilevel Programming Problem. *Nav. Res. Log.*, **35**, 413.
14. Floudas, C.A. and V. Visweswaran (1990). A Global Optimization Algorithm (GOP) for Certain Classes of Nonconvex NLPs-I. Theory. *Comput. Chem. Engng.*, **14**, 1397.
15. Floudas, C.A. and V. Visweswaran (1993). Primal-Relaxed Dual Global Optimization Approach. *JOTA*, **78**, 187.
16. Fortuny-Amat, J. and B. McCarl (1981). A representation and Economic Interpretation of a Two-Level Programming problem. *J. Oper. Res. Soc.*, **32**, 783.
17. Hansen, P. B. Jaumard and G. Savard (1990). New Branching and Bounding Rules for Linear Bilevel Programming. *SIAM J. Sci. Stat. Comp.*
18. Haurie, A., G. Savard and D.J. White (1990). A Note on: An Efficient Point Algorithm For a Linear Two-Stage Optimization Problem. *Oper. Res. Tech. Notes*, 553.
19. Judice J.J. and A.M. Faustino (1992). A sequential LCP method for Bilevel Linear Programming. *Annals Oper. Res.*, **34**, 89.
20. Shimizu, K. and E. Aiyoshi, E (1981). A New Computational Method for Stackelberg and Min-Max Problems by Use of a Penalty Method. *IEEE Trans. on Aut. Control*, **AC-26**, 460.
21. Tuy, H., A. Migdalas and P. Varbrand (1993a). A Global Optimization Approach for the Linear Two-Level Program. *J. Global Opt.*, **3**, 1.

22. Tuy, H., A. Migdalas and P. Varbrand (1993b). A Quasiconcave Minimization Method for Solving Linear Two-Level Programs. *J. Global Opt.*, 4, 243.
23. White D.J. and G. Anandalingam (1993). A Penalty Function Approach for Solving Bi-Level Linear Programs. *J. Global Opt.*, 3, 397.