NEW RESULTS IN THE PACKING OF EQUAL CIRCLES IN A SQUARE

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Abstract

The problem of finding the maximum diameter of n equal mutually disjoint circles inside a unit square is addressed in this paper. Exact solutions exist for only $n = 1, \ldots, 9, 10, 16, 25, 36$ while for other n only conjectural solutions have been reported. In this work a max–min optimization approach is introduced which matches the best reported solutions in the literature for all $n \leq 30$, yields a better configuration for $n = 15$, and provides new results for $n = 28$ and 29.

1 Introduction

The problem of finding the maximum diameter of equal non–overlapping circles contained in a unit square is equivalent to maximizing the minimum pairwise distance among n points in a unit square. This problem has been solved exactly for only $n = 1, \ldots, 9, 10, 16, 25, 36$. For $2 \leq n \leq 5$ the problem can be easily solved using simple geometric arguments. Graham derived the result for $n = 6$ according to Croft *et al.* [1], and Schaer [2, 3] first reported the solutions for $n = 8$ and $n = 9$.

The case $n = 10$ has been improved successively by Goldberg [4] and Schaer [5], However, the currently best known solution has first been reported by Schlüter [6]. Subsequently, Milano [7] and Valette [8] came up with less dense solutions and lately the best configuration has been published again independently by Grünbaum [9] and Mollard and Payan [10]. Recently, de Groot *et al.* [11] by using an elimination algorithm proved that the solution first given by Schlüter [6] is indeed exact.

The most thorough work on this problem has been published by Goldberg [4] in which conjectural optimal arrangements were provided for $n \leq 27$ as well as for some $n > 27$. For $n = 11$ and $n = 13$ Mollard and Payan [10] lately reported better solutions, for $n = 14$ first Wengerodt [12] and then Mollard and Payan [10] provided the same improved solution.

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Finally, for $n = 16, 25$ and 36, Wengerodt [13, 14, 15] matched the solutions given by Goldberg [4].

In this paper a max–min optimization approach is presented which yields improved solutions for $n = 15$, new configurations for $n = 28$ and 29 and matches the best reported configurations for up to $n = 30$.

2 Basic approach

The problem of maximizing the minimum pairwise distance of n points which are contained in a unit square can be formulated as the following max–min optimization problem.

$$\max_{xi,yi} \min_{(i,j)\in P} \quad s_{ij}$$

subject to

$$\begin{aligned}
&(x_i - x_j)^2 + (y_i - y_j)^2 = s_{ij}, \quad \forall\,(i,j) \in P \\
&0 \le x_i \le 1, \qquad i = 1, \ldots, N \\
&0 \le y_i \le 1, \qquad i = 1, \ldots, N
\end{aligned} \qquad \textbf{(P1)}$$

the set P is defined as $P = \{\,(i,j)$ such that $i < j\}$; x_i, y_i are the Cartesian coordinates of the i^{th} point; and s_{ij} is the squared Euclidean distance between the points i and j. The max–min optimization problem **(P1)** is equivalent to the following non–linear programming problem **(P2)**.

$$\max_{xi,yi} \quad t$$

subject to

$$\begin{aligned}
&(x_i - x_j)^2 + (y_i - y_j)^2 \ \ge\ t, \quad \forall\,(i,j) \in P \\
&0 \le x_i \le 1, \qquad i = 1, \ldots, N \\
&0 \le y_i \le 1, \qquad i = 1, \ldots, N
\end{aligned} \qquad \textbf{(P2)}$$

where t is the minimum over all the squared Euclidean distances s_{ij}. Formulation **(P2)** involves a linear objective function subject to quadratic concave inequality constraints plus box constraints for the x_i, y_i variables. By utilizing the General Algebraic Modeling System **GAMS 2.25** [16] as a programming environment, and the nonlinear programming solver **MINOS 5.3** [17] the programming problem **(P2)** was solved for every n up to $n = 30$. Since the employed solver provides no theoretical guarantee that the algorithm will converge to the global optimum, multiple initial points were used in order to span most of the parameter space. The selection of the initial points was based upon (i) partitioning the initial square into a number of equal rectangles whose sides are equal or almost equal and (ii) generating randomly points uniformily distributed inside every rectangle. For $n = 15, 28, 29$ new better configurations were found, in any other case the best reported solution was generated along with a plethora of slightly inferior configurations with differences in the fifth or even sixth decimal place in the objective function. The importance of this method lies in the fact that it provides an efficient way for systematically generating optimal configurations. Asymmetric configurations, among which the best solution is likely to be found for large n, can be easily generated since no assumptions are introduced for the distribution of the points inside the square. It should also be noted that this method can be easily modified for other packing problems like packing of spheres in a cube or packing of circles in a ralilatelar triangle.

3 Discussion of results

For $n \leq 9$ our approach generated the already proven best configurations, which are illustrated in Goldberg [4] and Croft *et al.* [1]. The problem for $n = 10$ has received considerable attention and many optimal configurations have been published; Goldberg [4] ($m = 0.41666667$), Schaer [5] ($m = 0.41954209$), Milano [7] ($m = 0.42014346$), Valette [8] ($m = 0.42118970$) and the best by Schlüter [6] ($m = 0.42127954$), and later by de Groot *et al.* [11]. Here m is the ratio of the minimum distance between any two points (or centers of circles) over the side of the unit square. All these configurations along with the following; ($m = 0.41469035, m = 0.41543009, m = 0.41837401, m = 0.41953837, m = 0.42072498, m = 0.42117156, m = 0.42126800$) have been generated with the proposed method.

For $n = 11$ the following optimal arrangements have been reported: Goldberg [4] ($m = 0.39801158$), and the currently best reported by Mollard and Payan [10] as well as de Groot *et al.* [11], ($m = 0.39820731$). The proposed method yielded the previous solutions plus the following: ($m = 0.39801104, m = 0.39801082$). For $n = 12$ the symmetric solution is the best so far, Goldberg [4] ($m = 0.38873013$), this solution along with ($m = 0.38206940$) have been obtained. For $n = 13$ first Goldberg [4] reported an optimal arrangement with ($m = 0.35355339$) and then Mollard and Payan [10] the currently best known ($m = 0.36609601$) which was also independently generated by our method and by de Groot *et al.* [11].

For $n = 14$ the following two solutions have been reported: Goldberg [4] ($m = 0.34509206$) and the best so far by Wengerodt [12] ($m = 0.34891526$). These configurations along with a large number of nearly optimal solutions have been generated. For $n = 15$ an improved configuration ($m = 0.34108138$), over the best so far ($m = 0.33860952$) derived by de Groot *et al.* [11], has been obtained. For $n = 16$ the optimum symmetric solution ($m = 0.33333333$) was generated which was first published by Goldberg [4].

For $n = 17$ three solutions were derived ($m = 0.30602129, m = 0.30611982, m = 0.30615399$) which match the first three decimal places of the best reported so far by Croft *et al* [1] ($m = 0.306\ldots$). For $n = 18, 19$, and 20 the currently best solutions published by Goldberg [4] ($n = 18,\ m = 0.30046261$), ($n = 19,\ m = 0.28954199$), and ($n = 20,\ m = 0.28661165$) have been matched. For $n = 21$ the best reported solution ($m = 0.272\ldots$) by Croft *et al* [1] is matched up to the third decimal place by the following configurations ($m = 0.27181169, m = 0.27181226, m = 0.27181675$).

For $n = 22$ an improved configuration was derived ($m = 0.26795840$), which was also reported by de Groot *et al.* [11], over the one obtained by Goldberg [4] ($m = 0.26794919$). For $n = 23, 24, 25, 27$ the best arrangements published by Goldberg [4] ($m = 0.258819045, m = 0.25433309, m = 0.25, m = 0.23584953$) respectively, were generated along with a larger number of slightly inferior solutions. For $n = 26$ the following configurations were obtained ($m = 0.23860970, m = 0.23872447, m = 0.23872458$) which match the solution reported by Croft *et al* [1] ($m = 0.239\ldots$). For $n = 28, 29$ the best derived configurations are ($m = 0.23053549, m = 0.22688290$) respectively, however, no comparisons can be made since no configurations were found in the literature. Finally, for $n = 30$ the best configuration reported by Goldberg [4] ($m = 0.22450296$) was generated. The values of m for all these solutions $n \leq 30$ are tabulated in Table 1. The adjacent graphs for the new solutions (n=15,28, and 29) are given in Figures 1, 2, 3. Due to space limitations further adjacent graphs as well as coordinates of the generated optimal configurations can be obtained by the authors upon request.

While the derived configurations correspond to local minima of formulation **(P2)**, the frequency with which they appear as solutions of **(P2)** suggest that they are reasonable candidates for being the global optimum solutions. Nevertheless, work is currently underway for applying a global optimization approach proposed by Floudas and Visweswaran [18, 19] in the problem at hand which will enable us to verify global optimality.

n	m	n	m	n	m
1		11	0.39820731	21	0.27181675
2	1.41421356	12	0.38873012	22	0.26795840
3	1.03527618	13	0.36609601	23	0.25881904
4	1.00000000	14	0.34891526	24	0.25433309
5	0.70710678	15	0.34108138	25	0.25000000
6	0.60092521	16	0.33333333	26	0.23872458
7	0.53589838	17	0.30615399	27	0.23584952
8	0.51763809	18	0.30046260	28	0.23053549
9	0.50000000	19	0.28954199	29	0.22688290
10	0.42127954	20	0.28661165	30	0.22450296

Table 1: Minimal separation between n points in a unit square.

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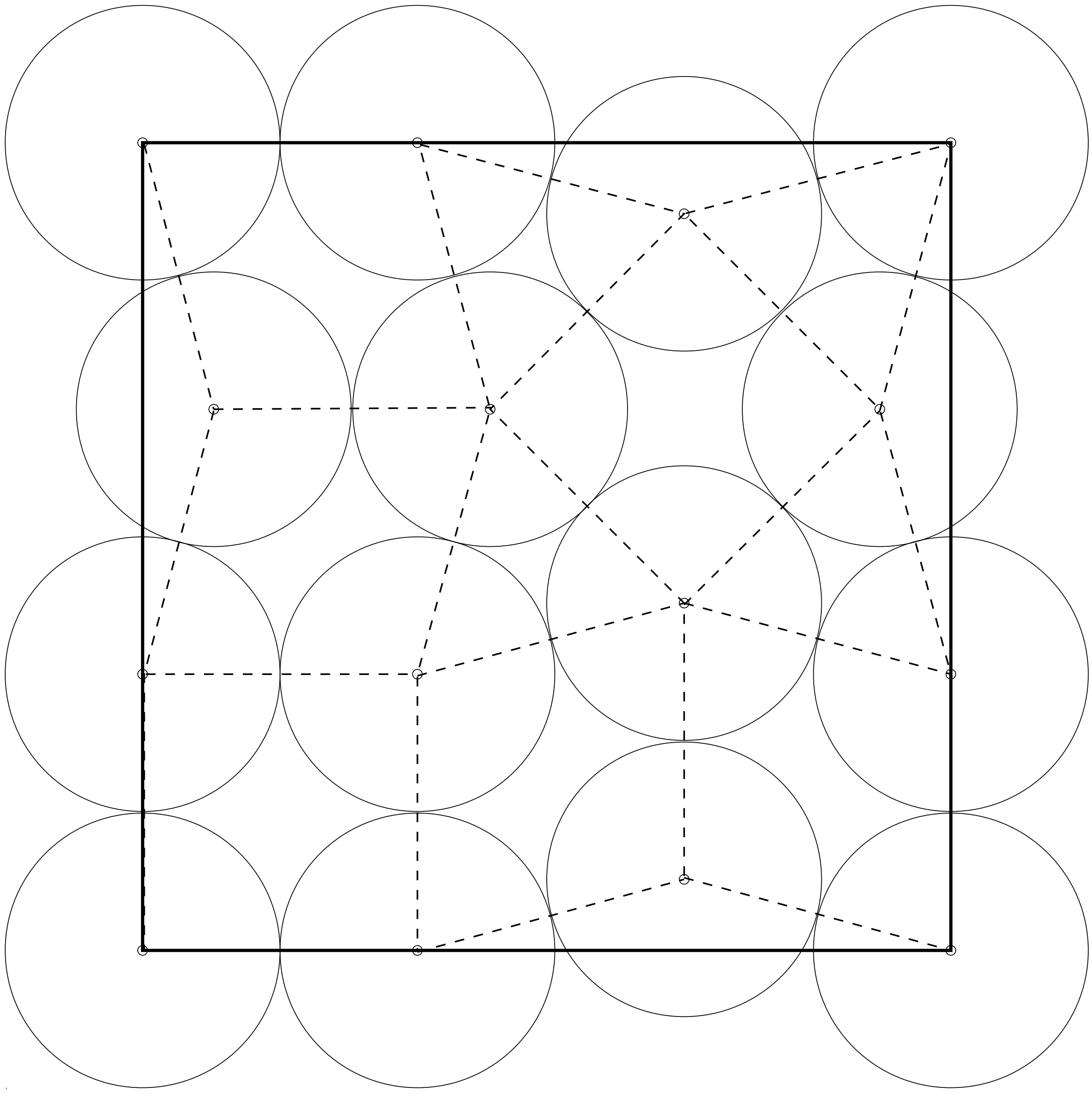

Figure 1: Adjacent graph for n=15, m=0.34108138

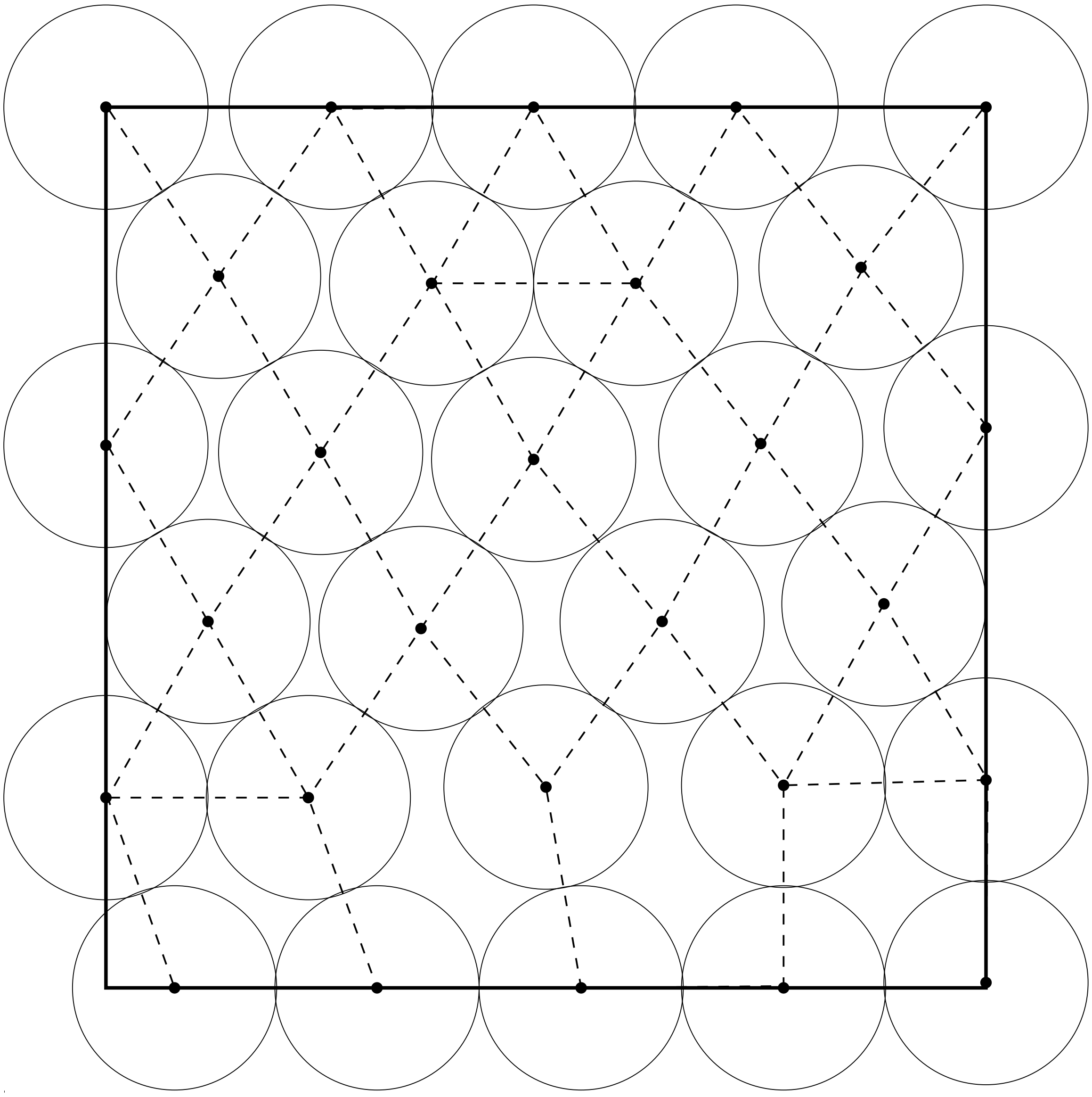

Figure 2: Adjacent graph for n=28, m=0.23053549

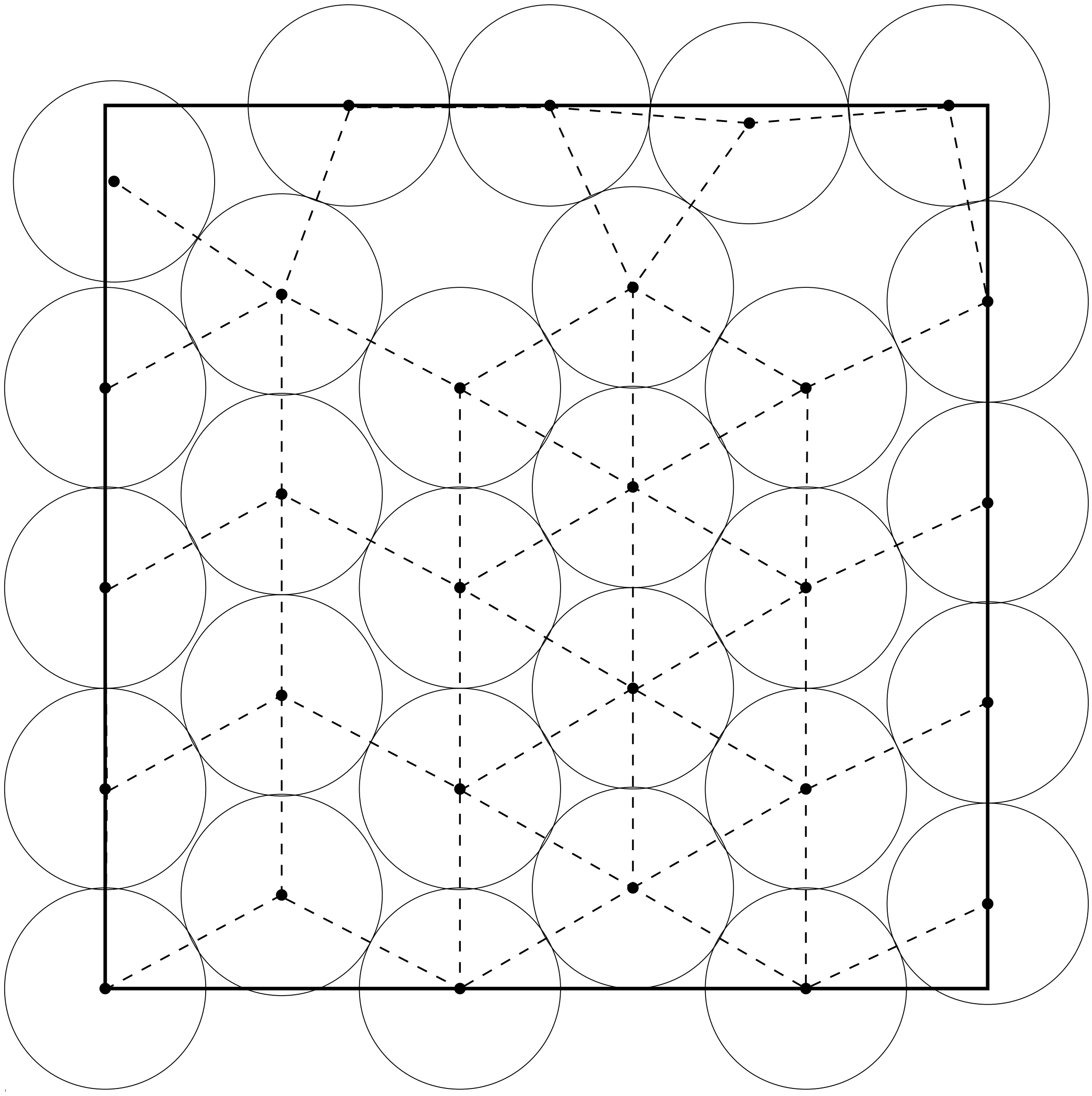

Figure 3: Adjacent graph for n=29, m=0.22688290